Series and Sequences:

A series is an attempt to add together an infinite list of numbers

$$a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

The individual numbers (such as a_3, a_7, a_{1003}) are called terms of the series. The sequence of terms is the entire list

$$\{a_1, a_2, a_3, a_4, \ldots\}$$

In general, a sequence is any infinite list of numbers

$$\{a_1, a_2, a_3, a_4, \ldots\} = \{a_n\}_{n=1}^{\infty}$$

Examples:

① $\{1, 4, 9, 16, 25, 36, 49, \ldots\} = \{n^2\}_{n=1}^{\infty}$

② $\{(n-7)^2\}_{n=8}^{\infty} = \{1, 4, 9, 16, 25, \ldots\}$

← This is an example of re-indexing

③ $\{\cos(n\pi/2)\}_{n=0}^{\infty} = \{1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots\}$

④ $\{k!\}_{k=0}^{\infty} = \{n!\}_{n=0}^{\infty} = \{1, 1, 2, 6, 24, 120, 720, \ldots\}$

note: By definition $0! = 1$

⑤ $\left\{\frac{(-1)^k}{k!}\right\}_{k=1}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, -\frac{1}{120}, \frac{1}{720}, -\frac{1}{5040}, \ldots\right\}$

⑥ Fibonacci Sequence $= \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots\}$
is generated by a "recursion": $a_n = a_{n-2} + a_{n-1}$.
(e.g.

A sequence $\{a_n\}_{n=1}^{\infty}$ has limit L, written as

$$\lim_{n\to\infty} a_n = L = \lim a_n$$

if the terms a_n get closer and closer to L as the integer n gets larger and larger. If the limit L exists and is finite, then the sequences <u>converges</u> and otherwise it <u>diverges</u>.

<u>Examples</u>:

① $\lim_{n\to\infty} \frac{1}{n} = 0$ converges

② $\lim_{n\to\infty} 5 = 5$ converges

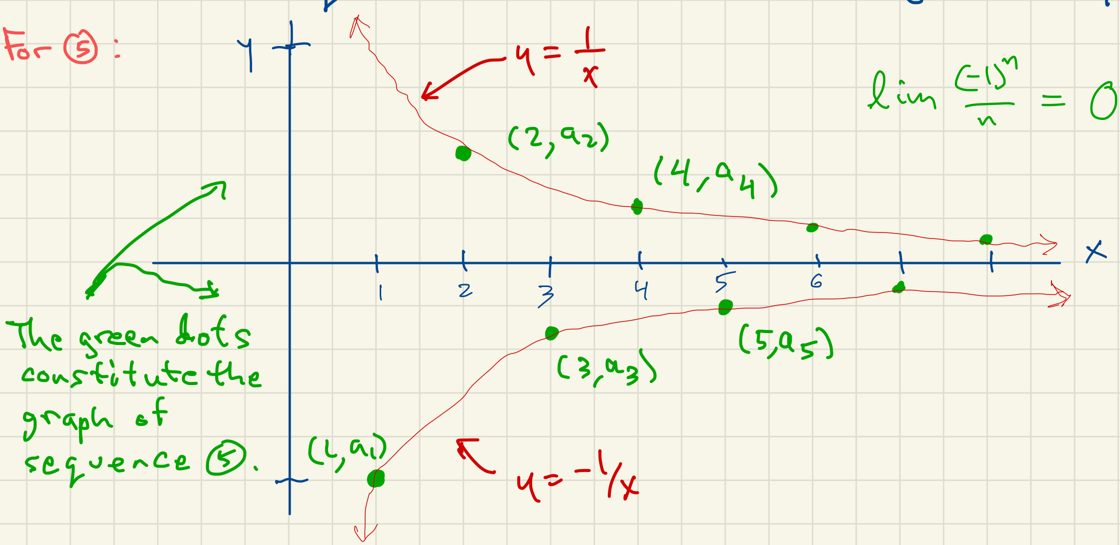
③ $\lim_{n\to\infty} \cos(n\pi/2) = \text{DNE}$ diverges

④ $\lim_{n\to\infty} k! = \infty$ diverges

⑤ $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ converges

The limit of a sequence can often be examined graphically.

For ⑤:



$$\lim \frac{(-1)^n}{n} = 0$$

Comments on examples 3 and 4.

3 $\{\cos(n\pi/2)\}_{n=0}^{\infty} = \{1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \dots\}$

$\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right) = \text{DNE}$ because the even terms get close to (actually equal) 0 for large values of n, but the terms a_n where n has remainder of 1 when divided by 4 get close to (actually equal) 1.

A sequence cannot have more than one limit.

4 Even though the sequence $\{k!\}$ diverges, we can write

$\lim_{k\to\infty} k! = \infty$ because $k!$ gets larger and larger as k gets larger and larger.

Useful Fact If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ for each integer n then $\lim_{n\to\infty} a_n = L$.

examples

1 $$\lim \frac{n^5+1}{3n^5-4n^4+2} = \lim \frac{1+\frac{1}{n^5}}{3-4\frac{1}{n}+2\frac{1}{n^5}} = \frac{1+0}{3+0+0} = \frac{1}{3}$$

divide top and bottom by n^5

2 $\lim \frac{n}{e^n} = 0$ (use L'Hopital's Rule)

$$\lim_{x\to\infty} \frac{x}{e^x} \underset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

∞/∞ form

$1/\infty$ form

Series

$$\sum_{n=1}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots$$

To determine the sum of the series we will consider the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ where

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

If $\lim_{n\to\infty} S_n = L$ then we say that L is the sum of the series $\sum_{n=0}^{\infty} a_n$.

If L is a finite number then the series converges otherwise it diverges.

What this is saying is:

To attempt to add infinitely many numbers $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ we will gradually add more and more terms together and see if the resulting sums get closer and closer to a number L.

examples:

① $\sum_{n=1}^{\infty} 1 = 1+1+1+1+1+\cdots = \infty$

because here we can find a formula for S_n:

$S_n = \text{sum of } n \text{ 1's} = n$ and $\lim_{n\to\infty} n = \infty$.

Since ∞ is not a finite number, the series $\sum_{n=1}^{\infty} 1$ diverges.

② $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$

Note that $s_1 = 1$, $s_2 = 1 + (-1) = 0$, $s_3 = 1$, $s_4 = 0$, $s_5 = 1$, ...

In general $s_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

The sequence of partial sums is

$$\{s_n\}_{n=1}^{\infty} = \{1, 0, 1, 0, 1, 0, 1, 0, \ldots\}$$

and $\lim_{n\to\infty} s_n = \text{DNE}$.

<u>Conclude</u> $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges (and $\sum_{n=1}^{\infty} (-1)^{n+1} = \text{DNE}$).

③ A "telescoping" example:

$$\sum_{n=1}^{\infty} \frac{1}{n+4} - \frac{1}{n+5} = \frac{1}{5} \qquad \underline{\text{converges!}}$$

Getting a nice formula for s_n is very rare!

Why? We can get a nice formula for s_n:

$$s_n = \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{n+4} - \frac{1}{n+5}\right)$$

$$= \frac{1}{5} - \frac{1}{n+5}$$ (because almost all terms subtract each other out.)

So the sequence of partial sums is

$$\{s_n\}_{n=1}^{\infty} = \left\{\frac{1}{5} - \frac{1}{n+5}\right\} \text{ and}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{5} - \frac{1}{n+5}\right) = \frac{1}{5}$$

④ A not-so-easy example:

There's no nice formula for s_n in this example.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

This series diverges (and in fact $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$)

(to be explained later in Chapter 11.)

⑤ A very difficult example: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$

Here $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

Later in chapter 11 will show that this series converges.

The sum $\pi^2/6$ was first discovered by Leonhard Euler, one of the all-time top 10 mathematicians. This result made him famous.

Quick Recap:

There are three important objects associated with a series:

(I) The series itself $\sum_{n=1}^{\infty} a_n$.

(II) The sequence of terms $\{a_n\}_{n=1}^{\infty}$.

(III) The sequence of partial sums $\{s_n\}_{n=1}^{\infty}$.

(where $s_n = s_1 + s_2 + s_3 + \cdots + s_n = \sum_{k=1}^{n} a_k$.)

$\boxed{BQ}$ **The Basic Question About Infinite Series:**

How can we tell whether or not $\sum_{n=1}^{\infty} a_n$ converges by just looking at the sequence of terms $\{a_n\}_{n=1}^{\infty}$??