For a real number x,

Answer True or False:

① If $x=3$ then $x^2=9$. True.

② If $x^2=9$ then $x=3$. False.

(because $(-3)^2=9$ but $-3 \neq 3$)

③ If $x=3$ or $x=-3$ then $x^2=9$. True.

④ $x=3 \Rightarrow x^2=9$ True.

(This is a restatement of ①)

⑤ $x^2=9 \iff x=\pm 3$ True.

(This is the same as saying that $\boxed{x^2=9 \Rightarrow x=\pm 3}$ and $\boxed{x=\pm 3 \Rightarrow x^2=9}$ are both true.)

More True/False Questions

① If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

True. This is the Test for Divergence.

② If $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges.

False !!

example

$\sum \frac{1}{n}$ diverges but $a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$.

③ If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

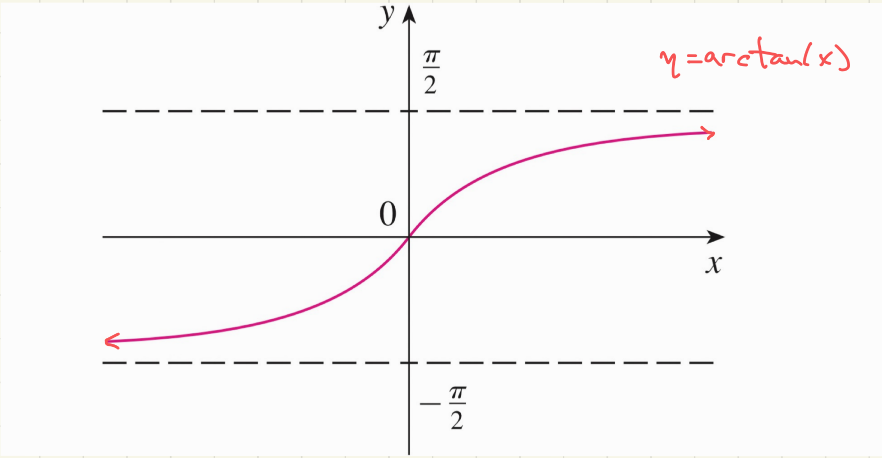
True. (This is another way to state the test for divergence.)

The arctan function

review

arctan(x) = angle θ between $-\pi/2$ and $\pi/2$ with $\tan(\theta) = x$

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$$



(notice horizontal asymptotes:

$\lim_{x\to\infty} \arctan(x) = \pi/2$, $\lim_{x\to-\infty} \arctan(x) = -\pi/2$)

example $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{n\sqrt{n}}$ Converge or Diverge??

$= \sum a_n$

$$a_n = \frac{\arctan(n)}{n\sqrt{n}} < \frac{\pi/2}{n\sqrt{n}} = \frac{\pi}{2}\,\frac{1}{n^{3/2}}$$

The series $\displaystyle\sum \frac{\pi}{2}\,\frac{1}{n^{3/2}} = \frac{\pi}{2}\sum \frac{1}{n^{3/2}}$ converges because

$\displaystyle\sum \frac{1}{n^{3/2}}$ is p-series with $p = 3/2 > 1$.

Conclude: $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{n\sqrt{n}}$ converges by CT.

(Comparison Test)

Factorials

For a positive integer n,

$$n! = n(n-1)(n-2)\cdots(3)\cdot(2)\cdot(1)$$

Thus $\boxed{(n+1)! = (n+1)\,n!}$

$\boxed{\text{And } 0! = 1}$

Also $(n+2)! = (n+2)(n+1)\,n!$

$$= (n^2+3n+2)\cdot n!$$

Examples

$$7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 7\cdot 6! = 7\cdot 720$$

$$7! = 5{,}040 = 5.04 \times 10^3$$

$$14! = 87{,}178{,}291{,}200 \approx 8.72 \times 10^{10}$$

Does $(2n)! \cancel{=} 2(n!)$? NO!!

The Ratio Test Let $\sum a_n$ be a positive series and

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

(i) If $L<1$ then $\sum a_n$ converges.

(ii) If $L>1$ then $\sum a_n$ diverges.

(iii) If $L=1$ this test is inconclusive.

example $\sum \frac{3^n n^2}{n!}$

$$a_n = \frac{3^n n^2}{n!} \qquad a_{n+1} = \frac{3^{n+1}(n+1)^2}{(n+1)!}$$

reciprocal of a_n

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}$$

$$= \frac{3^{\cancel{n}+1}}{\cancel{3^n}} \frac{(n+1)^2}{n^2} \frac{\cancel{n!}}{(n+1)\cancel{n!}} = 3\,\frac{n+1}{n^2} = 3\left(\frac{1}{n}+\frac{1}{n^2}\right)$$

So $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0. = L$

Conclude This series converges by ratio test.

A series which is not a positive series may have many terms that are negative and many that are positive. An important result that applies in this situation is:

Theorem (page 778 of Stewart)

Consider the series $\sum a_n$.

If the positive series $\sum |a_n|$ converges then $\sum a_n$ converges.

Terminology When $\sum |a_n|$ converges we will say that $\sum a_n$ converges absolutely.

Also If $\sum a_n$ converges but $\sum |a_n|$ does not converge then we say $\sum a_n$ converges conditionally.

example: $\sum \frac{(-1)^n 3^n n^2}{n!}$ converges absolutely

because

$$\left| \frac{(-1)^n 3^n n^2}{n!} \right| = \frac{3^n n^2}{n!} \quad \text{and} \quad \sum \frac{3^n n^2}{n!} \text{ converges.}$$

AST:

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

An Alternating Series has form $\sum a_n$ where the terms alternate between positive and negative.

example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

This is an alternating series with $b_n = \frac{1}{n}$ where

- $b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Conclude: $\sum \frac{(-1)^{n-1}}{n}$ converges by AST.

$\sum \frac{1}{n^p}$, $p=1$

Since $\left| \frac{(-1)^{n-1}}{n} \right| = \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges we can say $\sum \frac{(-1)^n}{n}$ converges conditionally.

Test your understanding of Convergence Tests to answer the Basic Question $\boxed{BQ}$ with these.

Exercises from Stewart : Some Easy — Some Not So Easy

should be able to apply indicated tests.

11.7 EXERCISES

1–38 Test the series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$ LCT

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ CT

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1}$ AST

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ DT

5. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ DT

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ Root

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ IT

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ Abs

9. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$ Ratio

10. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ IT

11. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3}+\frac{1}{3^n}\right)$ PS + GS

12. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ CT

13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ Ratio

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$ Abs + CT

15. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$ Ratio

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ LCT

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}$ Ratio

18. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ AST

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ AST

20. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$ LCT

21. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$ DT

22. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$ DT

23. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$ CT

24. $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$ DT

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ Ratio

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ LCT

27. $\displaystyle\sum_{k=1}^{\infty} \frac{k\ln k}{(k+1)^3}$ CT

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ CT

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$ AST

30. $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$ AST

31. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$ DT

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ root

33. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ Root

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ CT

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ LCT

36. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ LCT

37. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$ root

38. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$ LCT

NOTE: Many of these can be solved using more than one method.

See some notes on next page...

2. $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ (converge)

Can use LCT by comparing with $\sum b_n = \sum \frac{n}{n^3} = \sum \frac{1}{n^2}$

but CT would be quicker because $\frac{n-1}{n^3+1} < \frac{n-1}{n^3} < \frac{n}{n^3}$

8. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ (converge absolutely)

Here $\left| (-1)^{n-1} \frac{n^4}{4^n} \right| = \frac{n^4}{4^n}$ and $\sum \frac{n^4}{4^n}$ converges using

LCT with $\sum b_n = \sum \left(\frac{1}{4}\right)^n$ (or use ratio test)

11. $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n} \right)$ (converge)

$\sum \frac{1}{n^3}$ converges because it is the p-series for $p=3$

and $p>1$, $\sum \left(\frac{1}{3}\right)^n$ converges because it is geometric

series for $r = \frac{1}{3}$ which is between -1 and 1. Result

follows using linearity.

17. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)} = \sum a_n$ (converge)

If we write out $\frac{a_{n+1}}{a_n}$ carefully it simplifies to $\frac{2n+1}{3n+2}$

which limits to $2/3 < 1$ as $n \to \infty$. (using ratio test)

24. $\sum_{n=1}^{\infty} n \sin(1/n)$ (diverge) use L'Hopital to see that $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$.

34. $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$ (diverge) $\frac{1}{n+n\cos^2 n} \geq \frac{1}{2n}$ because $\cos^2 n \leq 1$.

and $\sum b_n = \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ diverges. (using CT)