There a few trig identities that are very useful for integration:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

These come from the

Addition formulas:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

$$\Rightarrow \quad \sin(2x) = \cos(x)\sin(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

What procedure can be used to solve?

$$\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$$

① $$\int \sin^3(x)\,dx = \int \sin^2 x \cdot \sin x \, dx = \int (1-\cos^2 x)\sin x \, dx$$

$$= -\int (1-u^2)\,du = -u + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C$$

$$\sin^2 x + \cos^2 x = 1$$

② $$\int \sin^4(x)\,dx = \int (\sin^2 x)^2\,dx = \int \left(\tfrac{1}{2}(\cos 2x - 1)\right)^2 dx$$

$$= \frac{1}{4}\int \cos^2(2x) - 2\cos(2x) + 1 \, dx$$

$= \ldots$ (see notes from last class)

③ $$\int \sin^3(x)\cos^3(x)\,dx = \int \sin^3 x \cos^2 x \cos x \, dx$$

$$= \int \sin^3 x \,(1-\sin^2 x)\cos x \, dx$$

$$\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$$

$$= \int u^3 (1-u^2)\,du$$

$$= \int u^3 - u^5 \, du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$

These techniques apply in theory to work any integral of the form

$$\int \sin^n(x)\cos^m(x)\,dx$$

where m and n are non-negative integers. But

example $\int \sin(x)^{1/2}\,dx$ cannot be worked in "closed form" — that is this integral does not equal a function that can be expressed using just the elementary functions that we have described and use in this course.

In other settings it is not uncommon to enlarge the class of "elementary functions" — for example "Bessel functions" are very useful in engineering math...

example $\int \sin(x)^{1/2}\cos(x)\,dx = \int u^{1/2}\,du$

$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}\sin(x)^{3/2} + C$

$$\boxed{\begin{cases} u = \sin x \\ du = \cos x\,dx \end{cases}}$$

Integrals of the form

$$\int \tan^n(x)\sec^m(x)\,dx$$

where m and n are non-negative integers can also be worked

The identity $\tan^2 - +1 = \sec^2\theta$ is very useful for this.

example $\int \tan^2(x)\sec^2(x)\,dx$ $\begin{cases} u = \tan x \\ du = \sec^2 x\,dx \end{cases}$

$$= \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

example $\int \tan^2(x)\sec(x)\,dx$ $\boxed{\tan^2 x = \sec^2 x - 1}$

$$= \int (\sec^2(x) - 1)\sec(x)\,dx$$

$$= \int \sec^3(x) - \sec(x)\,dx$$

done last class

$$= \left(\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x|\right) - \ln|\sec x + \tan x| + C$$

$$= \frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$$

example $\int \frac{\sin^2 x}{\cos^3 x}\,dx = \int \sin^2(x)\cos^{-3}(x)\,dx$

$$= \int \frac{\sin^2 x}{\cos^2 x}\,\frac{1}{\cos x}\,dx$$

$$= \int \tan^2(x)\sec(x)\,dx = \text{just worked this!!}$$

Problems from Stewart

7.5 EXERCISES

1–82 Evaluate the integral.

1. $\displaystyle\int \frac{\cos x}{1 - \sin x}\,dx$

2. $\displaystyle\int_0^1 (3x+1)^{\sqrt{2}}\,dx$

3. $\displaystyle\int_1^4 \sqrt{y}\ln y\,dy$

4. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

5. $\displaystyle\int \frac{t}{t^4+2}\,dt$

~~6.~~ $\displaystyle\int_0^1 \frac{x}{(2x+1)^3}\,dx$

7. $\displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy$

8. $\displaystyle\int t\sin t\cos t\,dt$

~~9.~~ $\displaystyle\int_2^4 \frac{x+2}{x^2+3x-4}\,dx$

10. $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx$

~~11.~~ $\displaystyle\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

~~12.~~ $\displaystyle\int \frac{2x-3}{x^3+3x}\,dx$

13. $\displaystyle\int \sin^5 t\cos^4 t\,dt$

~~14.~~ $\displaystyle\int \ln(1+x^2)\,dx$

15. $\displaystyle\int x\sec x\tan x\,dx$

~~16.~~ $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$

17. $\displaystyle\int_0^\pi t\cos^2 t\,dt$

18. $\displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

19. $\displaystyle\int e^{x+e^x}\,dx$

20. $\displaystyle\int e^2\,dx = e^2x + C$

~~21.~~ $\displaystyle\int \arctan\sqrt{x}\,dx$

22. $\displaystyle\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$

Some Hints

#1: multiply by $(1+\sin x)/(1+\sin x)$. (see next page)

#2: $\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C$. Take $p = \sqrt{2}$.

#3: Try IBP with $u = \ln(y)$

#4: $\dfrac{\sin^3(x)}{\cos(x)} = \dfrac{1-\cos^2(x)}{\cos(x)}\cdot\sin(x)$

#5: substitute $u = t^2$ and observe $t^4 = u^2$

#7: $\dfrac{d}{dy}[\arctan(y)] = ?$

#8: IBP

#10: $u = 1/x$ (see next page)

#13: $\sin^5 t = (1-\cos^2 t)^2\sin t$

#17: Try IBP with $u = t\cos t$

#19: $e^{x+e^x} = e^{e^x}e^x$ take $u = e^x$

#20: easiest of all ✓

#22: substitute $u = 1 + \ln(x)^2$

#1: $\frac{\cos(x)}{1-\sin(x)} \cdot \frac{1+\sin(x)}{1+\sin(x)} = \frac{\cos(x)+\sin(x)\cos(x)}{1-\sin^2(x)}$

$$= \frac{\cos(x)+\sin(x)\cos(x)}{\cos^2(x)}$$

$$= \frac{\cos(x)}{\cos^2(x)} + \frac{\sin(x)\cos(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)} = \sec(x) + \tan x$$

So $\int \frac{\cos x}{1-\sin x}dx = \int \sec(x) + \tan(x)\,dx =$

$= \ln|\sec x + \tan x| - \ln|\cos x| + C$

#10: Substitute $u = 1/x$, $du = -\frac{1}{x^2}dx$

and observe that $\frac{1}{x^3}dx = \left(-\frac{1}{x}\right)\left(-\frac{1}{x^2}dx\right) = -u\,du$.

Then $\int \frac{\cos(1/x)}{x^3}dx = -\int u\cos(u)\,du$ now use IBP.

#31: $\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{1+x}{1-x} \cdot \frac{1+x}{1+x}}$

$$= \sqrt{\frac{(1+x)^2}{1-x^2}} = \frac{1+x}{\sqrt{1-x^2}}$$

substitute
$\begin{cases} u = 1-x^2 \\ du = -2x\,dx \end{cases}$

So $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx$

$= \sin^{-1}(x) - \sqrt{1-x^2} + C$

23. $\int_0^1 (1+\sqrt{x})^8\,dx$

24. $\int (1+\tan x)^2 \sec x\,dx$

25. $\int_0^1 \frac{1+12t}{1+3t}\,dt$

26. $\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx$

27. $\int \frac{dx}{1+e^x}$

28. $\int \sin\sqrt{at}\,dt$

29. $\int \ln\left(x+\sqrt{x^2-1}\right)dx$

30. $\int_{-1}^{2} |e^x-1|\,dx$

31. $\int \sqrt{\frac{1+x}{1-x}}\,dx$

32. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

33. $\int \sqrt{3-2x-x^2}\,dx$

34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx$

35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1+\cos^2 x}\,dx$

36. $\int \frac{1+\sin x}{1+\cos x}\,dx$

37. $\int_0^{\pi/4} \tan^3\theta\,\sec^2\theta\,d\theta$

38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta\,\cot\theta}{\sec\theta}\,d\theta$

39. $\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta-\sec\theta}\,d\theta$

40. $\int_0^{\pi} \sin 6x\cos 3x\,dx$

41. $\int \theta\tan^2\theta\,d\theta$

42. $\int \frac{\tan^{-1}x}{x^2}\,dx$

43. $\int \frac{\sqrt{x}}{1+x^3}\,dx$

44. $\int \sqrt{1+e^x}\,dx$

45. $\int x^5 e^{-x^3}\,dx$

46. $\int \frac{(x-1)e^x}{x^2}\,dx$

47. $\int x^3(x-1)^{-4}\,dx$

48. $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

Hints

#24: $(1+\tan x)^2 = 1+2\tan x+\tan^2 x$
$= 2\tan x+\sec^2 x$

#30: $|e^x-1| = \begin{cases} e^x-1 & \text{if } x\geq 1 \\ 1-e^x & \text{if } x<1 \end{cases}$

#31: multiply inside square root by $\frac{1+x}{1+x}$
(see previous page)

#34: $\frac{1+4\cot x}{4-\cot x} = \frac{1+4\frac{\cos x}{\sin x}}{4-\frac{\cos x}{\sin x}} = \frac{\sin x+4\cos x}{4\sin x-\cos x}$
what's derivative of denominator?

#36: multiply $\frac{1-\cos x}{1-\cos x}$

#37 $\tan^2\theta\sec\theta = \sec^3\theta-\sec\theta$

#38,39: rewrite in terms of sin and cos

#40: $\sin(6x) = 2\sin(3x)\cos(3x)$

#45: substitute $u=-x^3$ and replace x^3 with $-u$.

Remark #35 The integrand $f(x) = \frac{x}{1+\cos^2 x}$ is an odd function, so $\int_{-a}^{a} f(x)\,dx = 0$ for any number a.

examples

Stewart pp 547–548

① $\int \frac{x+\arcsin(x)}{\sqrt{1-x^2}}\,dx$ #73

$= \int \frac{-2x}{-2\sqrt{1-x^2}} + \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$

$\begin{cases} u = 1-x^2 \\ du = -2x\,dx \end{cases}$

$= -\frac{1}{2}\int \frac{du}{u^{1/2}} + \int v\,dv$

$\begin{cases} v = \arcsin x \\ dv = \frac{1}{\sqrt{1-x^2}}\,dx \end{cases}$

$= -\frac{1}{2}\int u^{-1/2}\,du + \int v\,dv$

$= -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + \frac{v^2}{2} + C$

$= -\sqrt{1-x^2} + \frac{1}{2}\arcsin(x)^2 + C$

differentiate this to check answer !!

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

(2) $\int x \sin^2 x \cos x \, dx$ #79 IBP

$\begin{cases} u = x & du = dx \\ dv = \sin^2 x \cos x \, dx & v = \int \sin^2 x \cos x \, dx = \frac{1}{3}\sin^3(x) \end{cases}$

$= x \frac{1}{3}\sin^3(x) - \int \frac{1}{3}\sin^3(x)\, dx$

this integral was worked at start of class.

$= \frac{x}{3}\sin^3(x) + \cos(x) - \frac{1}{3}\cos^3(x) + C$

(3) $\int \frac{\sec x \cos 2x}{\sin x + \sec x} \, dx$ #80

$\frac{\sec(x)\cos(2x)}{\sin(x)+\sec(x)} = \frac{\frac{1}{\cancel{\cos(x)}}\cos(2x)}{\sin(x) + \frac{1}{\cos(x)}}$ $\frac{\cos\cancel{(x)}}{\cos\cancel{(x)}}$

$= \frac{\cos(2x)}{\sin(x)\cos(x)+1} = \frac{\cos(2x)}{\frac{1}{2}\sin(2x)+1}$

$\boxed{2\sin x \cos x = \sin 2x}$

$\int \frac{\cos(2x)}{\frac{1}{2}\sin(2x)+1}\, dx$ $\leftarrow$ $\begin{cases} u = \frac{1}{2}\sin(2x)+1 \\ du = \cos(2x)\, dx \end{cases}$

$= \int \frac{1}{u}\, du = \ln\left|\frac{1}{2}\sin(2x)+1\right| + C$