

Provarieties and observable subgroups of pro-affine algebraic groups

Andy R. Magid
Department of Mathematics
University of Oklahoma
Norman OK 73019

Nazih Nahlus
Department of Mathematics
American University of Beirut
Beirut, Lebanon

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Abstract

We define a closed subgroup H of a pro-affine algebraic group G to be *observable* if every (finite dimensional) rational H module is an H submodule of a rational G module. We show the equivalence to observability of a number of other conditions on the pair G, H , including the condition that the provariety G/H be pro-quasi-affine. To treat this latter, we also include results on the foundations of provarieties.

1 Introduction

Let K be an algebraically closed field, let G be an affine algebraic group over K and let H be a (Zariski) closed subgroup of G . Let V be a rational H module. Whether V occurs as an H -submodule of a rational G module W is a natural question and often an important technical step in applications. When every V so occurs, H is termed an *observable* subgroup of G . Observable subgroups were investigated by Bialynicki-Birula, Hochschild, and Mostow [1]; and by Grosshans [2],[3]; we recall and summarize their main result below (Remark (1)). (Some recent investigations of observability are found in [9] and [13].) Along with the various algebraic conditions equivalent to observability, one also has the geometric condition that H is observable in G if and only if G/H is a quasi-affine variety.

Our purpose in this work is to investigate the condition of observability for a closed subgroup H of a pro-affine algebraic group G .

The theory of pro-affine algebraic groups over K arises from the correspondence between (commutative) Hopf algebras over K and their groups of K points. The keys in the correspondence between Hopf algebras and pro-affine algebraic groups are the facts that inclusions of Hopf algebras always give rise to surjections on K points, that Hopf algebras are rings of K valued functions on their K points, and that surjections of Hopf algebras (or any commutative K algebras) give rise to inclusions of K points.

For a pro-affine algebraic group G over K with coordinate ring $K[G]$, the identification of G with the set $\text{Alg}_K(K[G], K)$ of K algebra homomorphisms from $K[G]$ to K , or K points of $K[G]$, also identifies G set theoretically as the inverse limit of the affine algebraic groups $G_i = \text{Alg}_k(A_i, k)$, where A_i ranges over the finitely generated K Hopf subalgebras of $K[G]$. $K[G]$ is the directed union of the A_i so that the set of K points of $K[G]$ is the inverse limit of the set of K points of the A_i .

The theory of pro-affine algebraic groups, their closed subgroups, and their coordinate rings and rational modules, is sufficient to obtain the various algebraic equivalent conditions to observability, which we do in Section (2) below.

It is the identification $G = \varprojlim G_i$ which gives G its pro-affine structure. We note, however, that this structure as just presented does not arise from viewing G as a group in a category of pro-affine varieties, although such an approach is possible. However the construction of a category of pro-affine varieties, or more generally of provarieties, is much more delicate than the correspondence between K Hopf algebras and pro-affine algebra groups. Unlike the situation with the latter, there are reduced K algebras with no K points, and hence which do not correspond in any meaningful way to pro-affine varieties (the statement in [12, p. 190] asserting such a correspondence is in error). For most purposes, in fact, it seems best to avoid the problem of the foundational problems of provarieties when dealing with pro-affine algebraic groups. For the investigations in this paper, however, the issue is unavoidable, since we also want to obtain a geometric condition on G/H for H being observable in the pro-affine group G , and G/H is not, in general, either an algebraic variety or a pro-affine algebraic group. In Section (3) below we provide our definition of provariety, which we then use to obtain our geometric condition equivalent to observability.

Notation and Conventions. We retain the notation K for our base field throughout. For a pro-affine algebraic group G over K we use $K[G]$ for its coordinate ring. If H is any (not necessarily closed) subgroup of G then $H' = K[G]^H$ and if S is any subset of a G module then $S' = \{g \in G \mid gs = s \ \forall s \in S\}$. If A is an integral domain contained in a field E we use $q.f.(A)$ to denote the quotient field of A inside E .

In the above notations, our main theorem is as follows:

Main Theorem. Let G be a connected pro-affine algebraic group over K and let H be an algebraic subgroup. Then the following are equivalent:

1. H is observable in G .
2. For every 1-dimensional rational H module that is an H submodule of some finite dimensional rational G module, the dual H module is also a submodule of a finite dimensional rational G module.

3. $q.f.(K[G]^H) = K(G)^H$.
4. $H = H''$.
5. $H = S'$ for some subset S of some rational G module.
6. $K[G]^H$ separates the elements of G/H .
7. G/H is a quasi-affine provariety. More precisely, there is a bijective morphism of provarieties from G/H (with its natural provariety structure) to a provariety $W = \varprojlim W_i$ where each W_i is a quasi-affine algebraic variety

We prove the equivalence of (1) to (6) in Section (2) (Theorem (3)) and add the equivalence of (6) and (7) in Section (3) (Theorem (5)). We also define and characterize quasi-affine provarieties in Section (3).

Remark 1. For the case when G is affine, the equivalent conditions of our main theorem are due to Bialynicki-Birula, Hochschild, and Mostow [1]; and Grosshans [2],[3].

For many of the equivalences, our proofs involve a reduction from the pro-affine to the affine case, and then an application of the results of [1] and [2],[3].

2 Observability

For general results on pro-affine algebraic groups we refer to [4], [5], [6], and [14].

Let G be a pro-affine algebraic group over K and let $\{A_i \mid i \in I\}$ be the set of finitely generated Hopf subalgebras of $K[G]$. For each $i \in I$ let G_i be the affine algebraic group with $K[G_i] = A_i$. Then $K[G] = \cup A_i$ and hence $G = \varprojlim G_i$. We refer to this description of G as the *standard limit* for G [11, p. 77]. Note that the transition maps $G_i \rightarrow G_j$ in the standard limit are all surjective.

Remark 2. Let $G = \varprojlim G_i$ be the standard limit for G . Then:

1. Each projection $p_i : G \rightarrow G_i$ is surjective. Let $K_i = \text{Ker}(p_i)$
2. Let $\{X_i \mid i \in I\}$ be a sub inverse system of $\{G_i \mid i \in I\}$ where each X_i is a left coset of an algebraic subgroup of G_i . Then $\varprojlim X_i$ is non-empty.
3. If $f : G \rightarrow P$ is a surjective morphism onto an affine algebraic group P , then for some i , f induces an isomorphism $G_i \rightarrow P$.

4. The Zariski topology on G is the inverse limit topology.

Proof. Both (1) and (2) follow from [4]. Since $f^*K[P]$ is a finitely generated Hopf subalgebra of $K[G]$, it coincides with some A_i , which proves (3). And (4) follows from Proposition (5) of Section (3) \square

Notation 1. Let $p_i : G \rightarrow G_i$ be as in Remark (2) (1). Let X be any subset of G . Then X_i denotes $p_i(X)$. (Note that the two meanings of G_i coincide since p_i is onto.)

Lemma 1. Let $G = \varprojlim G_i$ be the standard limit for G . Then:

1. If H is an algebraic subgroup of G then $H = \varprojlim H_i$. Moreover if $H = \varprojlim H_j$, $j \in J$, is the standard limit for H , then $\{H_i \mid i \in I\}$ is cofinal in $\{H_j \mid j \in J\}$.
2. Let $\{H_i\}$ be a sub inverse system of $\{G_i\}$ where each H_i is an algebraic subgroup of G_i . Then $H = \varprojlim H_i$ is an algebraic subgroup of G . Moreover, if the transition maps in the system $\{H_i\}$ are surjective, $p_i|_H$ is surjective.
3. If H is an algebraic subgroup of G then $K[G_i]^H = K[G_i]^{H_i}$. Moreover, if G is connected, then $K(G_i)^H = K(G_i)^{H_i}$.

Proof. By definition, the morphism $H \rightarrow H_i$ is surjective, and of course H_i is an algebraic subgroup of G_i . We identify $K[H_i]$ with its image in $K[H]$. Let $\rho : K[G] \rightarrow K[H]$ come from restriction, so $\rho^*(K[G_i]) = K[H_i]$. Since $K[G] = \cup K[G_i]$, we can apply ρ^* to see that $K[H] = \cup K[H_i]$, and this union is directed. It follows that $H = \varprojlim H_i$. It also follows that every finite subset of $K[H]$ lies in some $K[H_i]$. So if $H = \varprojlim H_j$ is the standard limit for H , each $K[H_j]$ lies in some $K[H_i]$. Hence (1).

Since $\{H_i\}$ is a sub inverse system of $\{G_i\}$, $\varprojlim H_i = \cap p_i^{-1} H_i$, and each $p_i^{-1}(H_i)$ is closed in G , so their intersection is as well, which makes the inverse limit a closed, hence algebraic, subgroup of G . If the transition maps are surjective, so are the projections by [4, Prop. 2.8, p. 505]. Hence (2).

As a subalgebra of $K[G]$, $K[G_i]$ is identified with $K[G]^{K_i}$. Hence $K[G_i]^H = K[G]^{K_i H} = (K[G]^{K_i})^{K_i H / K_i} = K[G_i]^{H_i}$. When G is connected, $K(G_i) = K(G)^{K_i}$, and then the same identities apply to fields of quotients. Hence (3). \square

It is a simple consequence of the above that representations of algebraic subgroups of G factor through finite dimensional quotients of G , a fact which we now record:

Lemma 2. Let H be an algebraic subgroup of G , let V be a finite dimensional rational H module and let $f : H \rightarrow \text{Aut}(V)$ be the associated morphism. Let $G = \varprojlim G_i$, $i \in I$, be the standard limit for G . Then there is $i \in I$ and a morphism $f_i : H_i \rightarrow \text{Aut}(V)$ such that $f = f_i \circ p_i$.

Proof. Since $f(H)$ is an affine algebraic subgroup of $\text{Aut}(V)$, we can apply Lemma (1) (1) and Remark (2) (2). \square

Lemma (2) allows one to reduce a number of results about observability immediately to the affine case: in the notation of the lemma, if V is an H_i submodule of a G_i module, this makes it an H submodule of a G module. We record a number of these in the following theorem:

Theorem 1. *Let H be an algebraic subgroup of the pro-affine algebraic group G .*

1. *If H is normal in G it is observable in G .*
2. *If H has pro-nilpotent radical (for example, if H is pro-reductive or pro-unipotent), then H is observable in G .*
3. *If G is pro-solvable then H is observable in G .*

Proof. As noted, we apply the corresponding results from [1] and [2],[3] for the affine case. \square

The following topological result, which will be needed for our main theorem, also has independent interest:

Theorem 2. *Let G be a pro-affine algebraic group over K . Then any family of closed left cosets of G with the finite intersection property has non-empty intersection.*

Proof. Let $\{S^t \mid t \in T\}$ be a family of closed left cosets of G with the finite intersection property. Let $G = \varprojlim G_i$ be the standard limit for G . Since the projections p_i map closed subgroups to algebraic, hence closed, subgroups, for each i , $\{p_i(S^t) = (S^t)_i\}$ is a family of closed left cosets, which has the finite intersection property since $\{S^t\}$ does. Let $X_i = \cap_t (S^t)_i$. Since G_i is Noetherian, X_i is a finite subintersection, and hence non-empty. It follows that X_i is itself a left coset of a subgroup of G_i . It is clear that $\{X_i\}$ is a sub inverse system of $\{G_i\}$, and so by Remark (2) (2) $X = \varprojlim X_i$ is non-empty. Let $x \in X$ and let $x_i = p_i(x)$. For given t , we have $x_i \in (S^t)_i$ for all i , and hence $x \in \varprojlim (S^t)_i$. Since S^t is closed in G , by Lemma (1) (1), $S^t = \varprojlim (S^t)_i$. Thus $x \in S^t$. Since this holds for every t , we conclude that $\cap_t S^t \neq \emptyset$. \square

As a consequence, we obtain the following analogue of [10, Satz 1]:

Corollary 1. *Let G be a proaffine algebraic group, let H be an arbitrary group, and let $f : G \rightarrow H$ be a group homomorphism. Assume that $\text{Ker}(f)$ is closed in G . Let $\{A_t \mid t \in T\}$ be a family of closed left cosets of G and assume that for every pair $s, t \in T$ there is $r \in T$ such that $A_r \subseteq A_s \cap A_t$. then $f(\cap A_t) = \cap f(A_t)$.*

Proof. Let $y \in \cap f(A_t)$. Since $\text{Ker}(f)$ is closed, $f^{-1}(y)$ is a closed left coset. Then the family $\{f^{-1}(y) \cap A_t \mid t \in T\}$ of (non-empty) closed left cosets has the finite intersection property and hence by Theorem (2) their intersection is non-empty. If z belongs to the intersection, then $z \in \cap A_t$ and $f(z) = y$, so $y \in f(\cap A_t)$ and thus $\cap f(A_t) \subseteq f(\cap A_t)$. The opposite inclusion is automatic and the result follows. \square

As noted in Remark (1), one condition equivalent to H observable in G in the affine case is that $H = H''$. Because G and H are finite dimensional, this is easily seen to be the same as $H = S'$ for some finite subset $S \subset K[G]$. We carry this condition over to the pro-affine case:

Definition 1. Let H be an algebraic subgroup of the pro-affine algebraic group G . Then H is super observable in G if there is a finite subset $S \subset K[G]$ such that $H = S' = \{g \in G \mid gs = s \ \forall s \in S\}$.

In this terminology we have:

Lemma 3. Let H be super observable in G and let $G = \varprojlim G_i$, $i \in I$ be the standard limit. Then there is $i_0 \in I$ such that for all $i \geq i_0$, H_i is observable in G_i . Consequently, H is observable in G .

Proof. Let $S \subset K[G]$ be finite such that $H = S'$. Choose i_0 such that $S \subset K[G_{i_0}]$. Let $i \geq i_0$. By Lemma (1) (3), $K[G_i]^H = K[G_i]^{H_i}$, from which it follows that $H_i = \{g \in G_i \mid gs = s \ \forall s \in S\}$. Then by Remark (1), H_i is observable in G_i , so every H_i module is an H_i submodule of a G_i module. Since this is true for every $i \geq i_0$, it follows from Lemma (2) that H is observable in G . \square

In final preparation for the proof of the algebraic cases of the main theorem, we record the following facts about closed subgroups of pro-affine groups:

Proposition 1. Let H be an algebraic subgroup of the pro-affine algebraic group G (connected when required). Then:

1. There is a subset E of $K[G]$ such that

$$H = \{g \in G \mid gs \in Ks \ \forall s \in E\}.$$

2. $H = q.f.(K(G)^H)'$

3. $K(G)^H$ separates the elements of G/H

Proof. Let $G = \varprojlim G_i$ be the standard limit for G . Select a subset E_i of $K[G_i]$ meeting condition (1) for H_i in G_i . Let $E = \cup E_i$. It is a straightforward application of Lemma (1) (3) to see that E satisfies (1) for H in G .

The same lemma also noted that $K(G_i)^H = K(G_i)^{H_i}$. Thus $K(G)^H = \cup K(G_i)^{H_i}$, and (2) follows from the similar assertion in the affine case.

Finally, (3) follows from Proposition (3) below. □

We can now prove the non-geometric parts of our Main Theorem:

Theorem 3. *Let G be a connected pro-affine algebraic group over K and let H be an algebraic subgroup. Then the following are equivalent:*

1. H is observable in G .
2. For every 1-dimensional rational H module that is an H submodule of some finite dimensional rational G module, the dual H module is also a submodule of a finite dimensional rational G module.
3. $q.f.(K[G]^H) = K(G)^H$.
4. $H = H''$.
5. $H = E'$ for some subset E of some rational G module.
6. $K[G]^H$ separates the elements of G/H .

Proof. The equivalence of (1) and (2) is identical with the affine case [1, Thm. 1]. The proof that (2) implies (3) is also word for word the same as the affine case [1, Thm. 4.3]. Now assume that $q.f.(K[G]^H) = K(G)^H$. Recall that we use $(\cdot)'$ to stand for the elements in G which act trivially on (\cdot) . We have $(K[G]^H)' \subseteq (q.f.(K[G]^H))'$ and under our assumption this latter is $(K(G)^H)'$. By Proposition (1) (2), $(K(G)^H)' = H$, so $(K[G]^H)' \subseteq H$. The reverse inclusion is trivial and hence we conclude $H'' = H$. Thus (3) implies (4). That (4) implies (5) is trivial. Now assume that $H = E'$, where E is a subset of some rational G module V . Let V^o be the K dual of V , as a rational G module. For $f \in V^o$ and $e \in E$ let $f/e \in K[G]$ be defined by $(f/e)(g) = f(ge)$. Let $V_E = \{f/e \mid f \in V^o, e \in E\} \subset K[G]$. One checks that in fact $V_E \subset K[G]^H$. We regard $K[G]^H$ as functions on G/H . Suppose that xH and yH are in G/H and suppose that $(f/e)(xH) = (f/e)(yH)$ for every $f/e \in V_E$. Then $xe = ye$ for all $e \in E$, so that $x^{-1}y \in E' = H$. Thus $xH = yH$. It follows that V_E , and hence $K[G]^H$, separates G/H . Thus (5) implies (6). Now suppose that $K[G]^H$ separates G/H , and let $g \in H'' = (K[G]^H)'$. For $f \in K[G]^H$ and $h \in H$ we have $f(gh) = (h \cdot f)(g) = f(g) = (g \cdot f)(1_G) = f(1_G)$, so that $f(gh) = f(H)$. Since by assumption $K[G]^H$ separates G/H , we have that $gH = H$, so that $g \in H$. Thus $H'' \subseteq H$. The reverse inclusion is trivial and hence $H = H''$. Thus (6) implies (4). To complete the equivalences, we assume that $H = H''$. Let V be

a finite dimensional rational H module and $f : H \rightarrow \text{Aut}(V)$ the corresponding representation. Let $G = \varprojlim G_i$, $i \in I$, be the standard limit for G . By Lemma (2), there is $i \in I$ and a morphism $f_i : H_i \rightarrow \text{Aut}(V)$ such that $f = f_i \circ p_i$. Let $\mathcal{S} = \{S^t \mid t \in T\}$ be the set of all the subgroups of G of the form E' where $E \subset H' = K[G]^H$ is finite. Note that $H = \cap S^t$, and that each S^t is a super observable subgroup of G in the sense of Definition (1). \mathcal{S} is closed under finite intersections, and it follows from Corollary (1) that $p_i(\cap S^t) = \cap p_i(S^t)$, hence $H_i = p_i(H) = p_i(\cap S^t) = \cap_t (S^t)_i$. Because G_i is Noetherian and \mathcal{S} is closed under finite intersections, for fixed i there is some $r \in T$ with $\cap_t (S^t)_i = (S^r)_i$. Thus $H_i = (S^r)_i$. Because S^r is super observable, by Lemma (3) there is $i_0 \in I$ such that for all $j \geq i_0$, $(S^r)_j$ is observable in G_j . We can take $j \geq i$. We regard the representation f_i of H_i as a representation of $(S^r)_i$ and hence as a representation of $(S^r)_j$. As such, V is an $(S^r)_j$ submodule of a G_j module W . We regard W as a G module via p_j , and then V becomes as H submodule of W . Hence (4) implies (1). \square

3 Provarieties

As shown above, a closed subgroup H of the pro-affine algebraic groups G is pro-affine, and in fact H is the inverse limit of subgroups H_i of the groups G_i defined above. It follows easily that G/H can be regarded as the (set theoretic) inverse limit of the inverse system of varieties G_i/H_i . We need to use this inverse limit structure to regard G/H as a provariety, and to make some general geometric comments about G/H based on this provariety structure.

We remark that provarieties appear in the literature in various contexts, including those that arise from inverse limits of algebraic groups. In addition to Peterson [12] who considers pro-affine algebraic groups, Kovacic [7] considers inverse limits of non affine algebraic groups, and Kowalski and Pillay [8] consider groups in the category of pro-algebraic varieties. Kovacic only considers groups which will not cover our case G/H when H is not normal. For Kowalski and Pillay, a provariety is an inverse system, not the inverse limit, which we want to use (and they restrict their inverse systems to countable ones). Therefore, we propose the following definition of provarieties:

Definition 2. Let $(X_\alpha, p_{\alpha,\beta})$, $\alpha \in \mathcal{A}$, $p_{\beta,\alpha} : X_\beta \rightarrow X_\alpha$ if $\beta \geq \alpha$ be an inverse system of irreducible algebraic varieties over K with directed index set \mathcal{A} and dominant transition maps. We denote the function field of X_α by $K(X_\alpha)$ and let $p_{\alpha,\beta}^* : K(X_\alpha) \rightarrow K(X_\beta)$ if $\beta \geq \alpha$. We let $K(X)$ denote $\varprojlim K(X_\alpha)$ and refer to it as the field of rational functions on X . We let p_α^* denote the canonical injection $K(X_\alpha) \rightarrow K(X)$. Then $X = \varprojlim X_\alpha$ is defined to be the ringed space (X, \mathcal{O}_X) defined as follows:

(a) X is the topological inverse limit of the spaces X_α . Let $p_\alpha : X \rightarrow X_\alpha$ denote the canonical projection. Basic open subsets of X are of the form $U = p_\alpha^{-1}(U_\alpha)$, where $U_\alpha \subseteq X_\alpha$ is open.

(b) Let $f \in K(X)$ and $x \in X$. Then f is said to be defined at x if there is a basic open set $U = p_\alpha^{-1}(U_\alpha)$ such that $x \in U$ and $f = p_\alpha^*(f_\alpha)$ for some $f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_{X_\alpha})$. We call U a defining neighborhood of f at x . Note that if U is a defining neighborhood for f , then f is defined at every point of U .

(c) Define the presheaf \mathcal{O}_X as follows: for $U \subset X$, let $\Gamma(U, \mathcal{O}_X) = \{f \in K(X) \mid f \text{ is defined at } x \forall x \in U\}$. For the empty set, we define $\Gamma(\emptyset, \mathcal{O}_X) = 0$. Note that, by definition, $\Gamma(U, \mathcal{O}_X) \subseteq K(X)$. If $U \subseteq V$ is an inclusion of open sets, $\Gamma(V, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$ denotes the inclusion for U nonempty, and the zero map otherwise.

We then have the following consequences of Definition (2)

Proposition 2. Let $X = \varprojlim X_\alpha$ be a provariety. The presheaf \mathcal{O}_X is a sheaf of K algebras and for $x \in X$ the stalk $\mathcal{O}_{X,x}$ is $\{f \in K(X) \mid f \text{ is defined at } x\}$.

Proof. First suppose that $f \in K(X)$ is defined at x and let $U = p_\alpha^{-1}(U_\alpha)$ be a basic open set such that $x \in U$ and $f = p_\alpha^*(f_\alpha)$ for some $f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_{X_\alpha})$. Note that if $U' = p_\beta^{-1}(U_\beta)$ is any basic open set with $x \in U'$ and if $\gamma \geq \alpha, \beta$ then for $U_\gamma = p_{\gamma,\beta}^{-1}(U_\beta) \cap p_{\gamma,\alpha}^{-1}(U_\alpha)$, $U'' = p_\gamma^{-1}(U_\gamma)$, and $f_\gamma = p_{\gamma,\alpha}^*(f_\alpha)$, we have $x \in U'' \subseteq U_\alpha \cap U_\beta$ and $f = p_\gamma^*(f_\gamma)$. It follows that if x is defined at x then any open set containing x contains a defining neighborhood of f . It further follows that f is determined, as an element of $K(X)$, by any defining neighborhood of any point of definition of f . Finally, this also implies that the rational functions defined at x form a K subalgebra of $K(X)$.

Now let $U \subseteq X$ be open and let $\{U^a \mid a \in A\}$ be a cover of U by nonempty basic open sets $U^a = p_{\alpha(a)}^{-1}(U_{\alpha(a)})$. We need to show that the sequence

$$\Gamma(U, \mathcal{O}_X) \rightarrow \prod_{a \in A} \Gamma(U^a, \mathcal{O}_X) \rightrightarrows \prod_{a,b \in A} \Gamma(U^a \cap U^b, \mathcal{O}_X)$$

is a coequalizer. The leftmost map is an injection, in fact for any $a \in A$ the map $\Gamma(U, \mathcal{O}_X) \rightarrow \Gamma(U^a, \mathcal{O}_X)$ is an injection. So suppose $(f^a)_{a \in A}$ belongs to the middle product and restricts to the same element of the rightmost product under both maps. It follows that as elements of $K(X)$, $f^a = f^b$ for all $a, b \in A$. Fix some $b \in A$, and let $f = f^b$ as an element of $K(X)$. If $x \in U$, and $x \in U^a$, then $f = f^a$ is defined at x , and it follows that $f \in \Gamma(U, \mathcal{O}_X)$ and that f has image $(f^a)_a$ under the leftmost map. Thus \mathcal{O}_X is a sheaf.

Finally, it is clear that if $x \in X$ then the stalk $\mathcal{O}_{X,x}$ consists of functions defined at x , since it is the direct limit (union) of subalgebras of $K(X)$ of functions defined at x . Conversely, if $f \in K(X)$ is defined at x , and U is any defining set for f at x , then $f \in \Gamma(U, \mathcal{O}_X)$, at the latter algebra of sections in contained in $\mathcal{O}_{X,x}$. □

We record the observation that the sections of \mathcal{O}_X in Proposition (2) over basic open sets have a natural direct limit structure, and hence that stalks are direct limits.

Lemma 4. Let $X = \varinjlim X_\alpha$ be a provariety and let $U = p_\alpha^{-1}(U_\alpha)$, where $U_\alpha \subseteq X_\alpha$ is open, be a basic open set. Then $p_\alpha^*(\Gamma(U_\alpha, \mathcal{O}_{U_\alpha})) \subseteq \Gamma(U, \mathcal{O}_X)$. Moreover if $f \in \Gamma(U, \mathcal{O}_X)$, then there is $\beta \geq \alpha$, an open $U_\beta \subseteq p_{\beta,\alpha}^{-1}(U_\alpha)$ and an $f_\beta \in \Gamma(U_\beta, \mathcal{O}_{U_\beta})$ such that $f = p_\beta^*(f_\beta)$ in $K(X)$. In particular, for $x \in X$

$$\mathcal{O}_{X,x} = \varinjlim p_\alpha^*(\mathcal{O}_{X_\alpha, p_\alpha(x)}).$$

□

Under some stronger hypotheses on the inverse system $(X_\alpha, p_{\alpha,\beta})$ we have the following strengthening of Lemma (4):

Lemma 5. Assume that in the inverse system $(X_\alpha, p_{\alpha,\beta})$, $\alpha \in \mathcal{A}$, $p_{\beta,\alpha} : X_\beta \rightarrow X_\alpha$, for $\beta \geq \alpha$ the transition maps are surjective and satisfy

$$\mathcal{O}_{X_\beta, x_\beta} \cap p_{\beta,\alpha}^*(K(X_\alpha)) = p_{\beta,\alpha}^*(\mathcal{O}_{X_\alpha, x_\alpha}) \text{ where } p_{\beta,\alpha}^*(x_\beta) = x_\alpha$$

Let $X = \varinjlim X_\alpha$ and suppose that the maps $X \rightarrow X_\alpha$ are all surjective. Let $U = p_\alpha^{-1}(U_\alpha) \subseteq X$ be basic open. Then

$$\Gamma(U, \mathcal{O}_X) = \varinjlim_{\beta \geq \alpha} p_\beta^*(\Gamma(p_{\beta,\alpha}^{-1}(U_\alpha), \mathcal{O}_{X_\beta}))$$

and in particular, $\Gamma(U, \mathcal{O}_X) \cap p_\alpha^*(K(X_\alpha)) = p_\alpha^*(\Gamma(p_\alpha(U), \mathcal{O}_{X_\alpha}))$.

Proof. It is clear from the definition of \mathcal{O}_X that the direct limit is contained in $\Gamma(U, \mathcal{O}_X)$. Let $f \in \Gamma(U, \mathcal{O}_X)$. By Lemma (4), for each $x \in U$ there is $\beta(x) = \beta \geq \alpha$ such that $f = p_\beta^*(f_\beta) \in p_\beta^*(\mathcal{O}_{X_\beta, p_\beta(x)})$. Fix one such β ; without loss of generality we may assume $\beta = \alpha$. Then for any $x \in U$ and for $\beta = \beta(x)$, $x_\beta = p_\beta(x)$, and $x_\alpha = p_\alpha(x)$ we have $p_{\beta,\alpha}^*(f_\alpha) = f_\beta \in \mathcal{O}_{X_\beta, x_\beta}$, and by assumption this implies that $f_\beta \in p_{\beta,\alpha}^*(\mathcal{O}_{X_\alpha, x_\alpha})$. But then $f_\alpha \in \mathcal{O}_{X_\alpha, x_\alpha}$. This holds for every $x \in U$, and hence $f_\alpha \in p_\alpha^*(\Gamma(U_\alpha, \mathcal{O}_{U_\alpha}))$, as desired.

Note that for this argument it is sufficient simply to have $f = p_\alpha^*(f_\alpha)$: for then f_α is defined at x_α for at some $x_\alpha \in U_\alpha$, and we can take $x \in p_\alpha^{-1}(x_\alpha)$. The final assertion of the lemma follows. □

The conclusion of Lemma (5) can be informally phrased as follows: under the hypotheses of the lemma, if a section of \mathcal{O}_X over a basic open set U comes from a rational function on $p_\alpha(U)$, then it comes from a section of \mathcal{O}_{X_α} over $p_\alpha(U)$.

We refer below to the hypotheses of Lemma (5) as the *transition map conditions*. The conditions will be satisfied when the inverse system of X_α 's arises from a pro-affine algebraic group $G = \varinjlim G_\alpha$ and a pro-affine algebraic subgroup H taking $X_\alpha = G_\alpha/H_\alpha$, where $H_\alpha = p_\alpha(H)$.

The following proposition analyzes when the global sections of a provariety separates points.

Proposition 3. Let $X = \varinjlim X_\alpha$ be a provariety and suppose the inverse system $(X_\alpha, p_{\alpha,\beta})$, $\alpha \in \mathcal{A}$, $p_{\beta,\alpha} : X_\beta \rightarrow X_\alpha$, for $\beta \geq \alpha$ satisfies all the transition conditions of Lemma (5). Assume that $\Gamma(X, \mathcal{O}_X)$ separates the points of X .

Then if $v_\alpha \neq w_\alpha$ are points of X_α ($\alpha \in \mathcal{A}$), there is $\beta \geq \alpha$ and points v_β, w_β of X_β with $p_{\beta,\alpha}(v_\beta) = v_\alpha$, $p_{\beta,\alpha}(w_\beta) = w_\alpha$ and $f_\beta \in \Gamma(X_\beta, \mathcal{O}_{X_\beta})$ such that $f_\beta(v_\beta) \neq f_\beta(w_\beta)$. Conversely, if for each α and each pair of points in X_α there is $\beta \geq \alpha$ and $f_\beta \in \Gamma(X_\beta, \mathcal{O}_{X_\beta})$ separating preimages of those points in X_β , then $\Gamma(X, \mathcal{O}_X)$ separates the points of X .

Proof. We begin with $v_\alpha \neq w_\alpha$, and choose $v, w \in X$ such that $p_\alpha(v) = v_\alpha$ and $p_\alpha(w) = w_\alpha$. By assumption, there is $f \in \Gamma(X, \mathcal{O}_X)$ such that $f(v) \neq f(w)$. By Lemma (5), there $f = p_\beta^*(f_\beta)$ for some β , which we can assume, without loss of generality, satisfies $\beta \geq \alpha$. Take $v_\beta = p_\beta(v)$ and $w_\beta = p_\beta(w)$; then $v_\beta, w_\beta, f_\beta$ meet the conclusions of the proposition. The final statement of the proposition follows from Lemma (5): namely, from

$$\Gamma(X, \mathcal{O}_X) = \varinjlim \Gamma(X_\beta, \mathcal{O}_{X_\beta}).$$

□

A possible stronger form of Proposition (3), which would assert that a provariety whose points are separated by their global sections is an inverse limit of varieties separated by their global sections (i.e. quasi-affine varieties), is false, as the following example shows:

Example 1. Let G denote $SL_n(\mathbb{C})$ and let B denote its Borel subgroup of upper triangular matrices, so $G/B = \mathbb{P}_1(\mathbb{C})$. We define an inverse system indexed by the natural numbers \mathbb{N} as follows: take $X_1 = G$. For $n > 1$, let $X_n = G^{(n-1)} \times G/B$. Define $p_{n+1,n}$ by projection onto the first n factors followed by the obvious surjection $G^{(n)} \rightarrow G^{(n-1)} \times G/B$. For $n > m$ we define $p_{n,m}$ by composition, and of course $p_{n,n}$ is the identity. The inverse system $\{X_n, p_{n,m} \mid n \in \mathbb{N}\}$ satisfies the transition conditions. Let $X = \varprojlim X_n$. It is clear from Proposition (3), or by a simple direct argument, that $\Gamma(X, \mathcal{O}_X)$ separates the points of X . On the other hand, it is also clear that none of the X_i 's are separated by their global sections.

We next want to consider the functorial properties of Definition (2). First, we define morphisms, beginning with a morphism of inverse systems:

Definition 3. Let $(X_\alpha, p_{\alpha,\beta})$, $\alpha \in \mathcal{A}$, $p_{\beta,\alpha} : X_\beta \rightarrow X_\alpha$ if $\beta \geq \alpha$ and $(Y_\gamma, p_{\gamma,\delta})$, $\gamma \in \mathcal{B}$, $q_{\delta,\gamma} : Y_\delta \rightarrow Y_\gamma$ if $\delta \geq \gamma$ be directed inverse systems of irreducible algebraic varieties over K with dominant transition maps. A morphism between the systems is a pair (ϕ, \mathcal{F}) where $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is an order preserving map with cofinal image and $\mathcal{F} = \{f_\alpha : X_\alpha \rightarrow Y_{\phi(\alpha)} \mid \alpha \in \mathcal{A}\}$ is a set of morphisms such that if $\beta \geq \alpha$ then $q_{\phi(\beta), \phi(\alpha)} f_\beta = f_\alpha p_{\beta,\alpha}$.

For example, if $(X_\alpha, p_{\alpha,\beta})$, $\alpha \in \mathcal{A}$, $p_{\beta,\alpha} : X_\beta \rightarrow X_\alpha$ is an inverse system as in Definition (3) and we fix $\gamma \in \mathcal{A}$, we can let $\mathcal{A}_0 = \{\gamma\}$ and consider X_γ as the inverse system over \mathcal{A}_0 with identity transition map. Then the constant map $\phi : \mathcal{A} \rightarrow \mathcal{A}_0$ is order preserving and has cofinal image. Let $\mathcal{F} = \{p_{\beta,\gamma} \mid \beta \geq \gamma\}$. It follows that (ϕ, \mathcal{F}) is a morphism from $(X_\alpha, \alpha \in \mathcal{A})$ to $(X_\gamma, \gamma \in \mathcal{A}_0)$.

A morphism of inverse systems defines a function between their set theoretic inverse limits. In the notation of definition (3), the function $F : X \rightarrow Y$ where $X = \varprojlim X_\alpha$ and $Y = \varprojlim Y_\gamma$ as sets as defined by the morphism (ϕ, \mathcal{F}) , and given by the conditions $f_\alpha p_\alpha = q_{\phi(\alpha)} F \forall \alpha \in \mathcal{A}$. Since $\phi(\mathcal{A})$ is cofinal in \mathcal{B} we can replace the latter by the former and assume ϕ is surjective. We continue to follow this notation.

Let $y \in Y$, let g be a rational function on Y defined at y , and let $V = q_\gamma^{-1}(V_\gamma)$, where $V_\gamma \subseteq Y_\gamma$ is open, be a defining set for g at y with $g = q_\gamma^*(g_\gamma)$. Suppose $F(x) = y$. Suppose $\gamma = \phi(\alpha)$. Let $U_\alpha = f_\alpha^{-1}(V_\gamma)$. Then $U_\alpha \subseteq X_\alpha$ is open and $U = p_\alpha^{-1}(U_\alpha)$ is basic open. From f_α we have the map $f_\alpha^* \Gamma(V_\gamma, \mathcal{O}_{Y_\gamma}) \rightarrow \Gamma(U_\alpha, \mathcal{O}_{X_\alpha})$. Thus $f = p_\alpha^* f_\alpha^* g_\gamma$ is, by Lemma (4), a rational function on X defined on U . Note that $x \in U$. We define $F_x^* : \mathcal{O}_{Y, F(x)} \rightarrow \mathcal{O}_{X, x}$ by $g \mapsto f$. (We omit the straight forward verifications that this map is well-defined and that it is a K algebra map.) We can then apply this map to every point of $F^{-1}(V)$ and obtain a map $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(F^{-1}(V), \mathcal{O}_X)$.

We can now show that morphisms of inverse systems can be used to define morphisms of provarieties:

Proposition 4. *Let $F : X = \varprojlim X_\alpha \rightarrow Y = \varprojlim Y_\gamma$ be a morphism of (the underlying sets of) provarieties arising from a morphism of inverse systems. Then if V is a basic open subset of Y then $F^{-1}(V)$ is a basic open subset of X , and hence F is continuous. There is a sheaf morphism $F^* \mathcal{O}_Y \rightarrow F_*(\mathcal{O}_X)$.*

Proof. The first assertion was noted in the discussion above. Recall that $F_*(\mathcal{O}_X)$ is the sheaf on Y whose section on the open subset V are given by $\Gamma(V, F_*(\mathcal{O}_X)) = \Gamma(F^{-1}(V), \mathcal{O}_X)$; then second assertion was also established above. \square

Proposition (4) says that the function $F : X \rightarrow Y$ on the underlying sets of the provarieties X and Y gives rise to a map ringed spaces. We extend the notation of the proposition in the usual way: for $V \subseteq Y$ open we have a K algebra map on sections

$$F_V^* : \Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(F^{-1}(V), \mathcal{O}_X)$$

and for $x \in X$ we have a K algebra map on stalks

$$F_x^* : \mathcal{O}_{Y, F(x)} \rightarrow \mathcal{O}_{X, x}.$$

If we combine the above formula for F_x^* with the final direct limit of Lemma (4), we have the following description of morphisms of provarieties on stalks:

Lemma 6. *Let $F : X = \varprojlim X_\alpha \rightarrow Y = \varprojlim Y_\gamma$ be a morphism of provarieties arising from the map (ϕ, \mathcal{F}) of inverse systems. Then for $x \in X$*

$$F_x^* = \varinjlim f_{\alpha, \phi(\alpha)} : \varinjlim q_{\phi(\alpha)}^* (\mathcal{O}_{Y_{\phi(\alpha)}, f_\alpha(p_\alpha(x))}) \rightarrow \varinjlim p_\alpha^* (\mathcal{O}_{X_\alpha, p_\alpha(x)}).$$

□

It is a consequence of our definition of provariety and of morphism that if $X = \varprojlim X_\alpha$ is a provariety then the projection $p_\alpha : X \rightarrow X_\alpha$ is a morphism of provarieties. We also have the following for closed subsets of provarieties:

Let $X = \varprojlim X_\alpha$ be a provariety. Suppose $Y \subseteq X$ satisfies

$$Y = \cap_\alpha p_{\alpha-1}(\overline{p_\alpha(Y)})$$

Then the morphism $\varprojlim \overline{p_\alpha(Y)}$ has image Y . We call such subsets of a provariety *saturated*, and the content of this remark is that we can regard saturated subsets Y of X as provarieties.

Saturated subsets are clearly closed. We remark that the converse is true as well:

Lemma 7. *Let Y be a closed subset of the provariety X . Then Y is saturated.*

Proof. Suppose $X = \varprojlim X_\alpha$ and $Y \subseteq X$ is closed. Let $x \in X - Y$, and let $U = p_\alpha^{-1}(U_\alpha)$ be a basic open subset with $x \in U$ and $U \cap Y = \emptyset$. If $z = p_\alpha(y) \in U_\alpha$ then $y \in U$, so $p_\alpha(Y) \cap U_\alpha = \emptyset$. Since U_α is open, this means that $\overline{p_\alpha(Y)} \cap U_\alpha$ is empty as well, and thus $p_{\alpha-1}(\overline{p_\alpha(Y)}) \cap U = \emptyset$, so in particular $x \notin p_{\alpha-1}(\overline{p_\alpha(Y)})$. So $x \notin Y$ implies $x \notin \cap_\alpha p_{\alpha-1}(\overline{p_\alpha(Y)})$, and the contrapositive of this in the assertion of the lemma. □

If $\{X_i \mid i \in I\}$ is any set of algebraic varieties, and A is any subset of I , then we let X_A denote $\prod_{i \in A} X_i$. The set \mathcal{F} of finite subsets of I is partially ordered by inclusion, and directed in this partial order. Set theoretically, we have

$$\prod_{i \in I} X_i = \varprojlim \{X_F \mid F \in \mathcal{F}\}.$$

We use this identification to view $\prod X_i$ as a provariety. Note that the topology on $\prod X_i$ is defined by basic open subsets U of the form $U_F \times \prod_{i \notin F} X_i$.

Proposition 5. *Let $(X_\alpha, p_{\alpha,\beta})$ be an inverse system of algebraic varieties with inverse limit X . Then the canonical image of X in $P = \prod X_\alpha$ is saturated and the canonical map $q : X \rightarrow P$ is a morphism of provarieties. Let $q(X)$ be given its provariety structure as a saturated subset. Then $X \rightarrow q(X)$ is an isomorphism of provarieties.*

Proof. The image $q(X)$ of X in P is $\{(x_\alpha) \mid x_\alpha = p_{\beta,\alpha}(x_\beta) \text{ for } \beta \geq \alpha\}$. Thus $q(X) = \cap_{\beta \geq \alpha} G_{\beta,\alpha} \times X_{I-\{\alpha,\beta\}}$ where $G_{\beta,\alpha} = \{(x_\beta, x_\alpha) \mid x_\alpha = p_{\beta,\alpha}(x_\beta)\}$ is the graph of $p_{\beta,\alpha}$. These graphs are all closed, of course. Suppose $F \subseteq \mathcal{A}$ is a finite set of indices. The projection $p_F(q(X))$ lies inside the subset X'_F of X_F consisting of consistent tuples $(x_a)_{a \in F}$. The same remark on graph closures implies that X'_F is closed in X_F , so that $X''_F = X'_F \times \prod_{\alpha \notin F} X_\alpha$ is closed in P . It is clear that $\overline{p_F(q(X))} \subseteq X'_F$ so that $p_F^{-1}(\overline{p_F(q(X))}) \subseteq X''_F$, and that $q(X) = \cap_F X''_F$. It follows that $q(X)$ is a saturated subset of P .

It remains to identify the provariety structure on $q(X)$ as a saturated subset of P with that of the given provariety structure on X . It is trivial that $X \rightarrow q(X)$

is a homeomorphism of the underlying topological spaces, and we already know it is a morphism of provarieties. To see that it is an isomorphism of provarieties, it suffices to consider the map on stalks, and then the result follows from the case of finite products by Lemma (6). \square

We will use Proposition (5) to identify the inverse limit X and the set of consistent tuples $q(X)$.

The combination of Proposition (5) and Lemma (6) provides a description of the stalks of provariety, and in the case of a pro-affine variety, of the coordinate ring:

Corollary 2. *Let $(X_\alpha, p_{\alpha,\beta})$ be an inverse system of algebraic varieties with inverse limit X , and let $x = (x_\alpha)$ belong to $X \subseteq \prod X_\alpha$. Then:*

1. *There is a surjection $\otimes_\alpha \mathcal{O}_{X_\alpha, x_\alpha} \rightarrow \mathcal{O}_{X, x}$*
2. *If the X_α are all affine, then there is a surjection $\otimes_\alpha \Gamma(X_\alpha, \mathcal{O}_{X_\alpha, x_\alpha}) \rightarrow \Gamma(X, \mathcal{O}_{X, x})$*

\square

Of course the surjections in Corollary (2) tacitly assume that X is non-empty. This need not be the case, even when all X_α are affine: for example, let $K(x)$ be the field of rational functions over K . We can write $K(x) = \varinjlim A_i$ where $\{A_i \mid i \in I\}$ ranges over the finitely generated K subalgebras of $K(x)$, and form the inverse system of affine varieties X_i where $\Gamma(X_i, \mathcal{O}_{X_i}) = A_i$. Then $\varprojlim X_i$ is empty (it corresponds to the K points of $K(x)$). In terms of Proposition (5) and its proof, this implies that the closed subsets X_F'' , which have the finite intersection property, have empty intersection. In particular, $\prod X_i$ is *not* quasi-compact.

On the other hand, suppose we begin with a set X and an integral domain A of K valued functions on X which separates the points of X . Suppose $\{A_\alpha, \alpha \in \mathcal{A}\}$ is a set of finitely generated K subalgebras of A indexed by the directed set \mathcal{A} , and such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$. Let $X_\alpha = \text{Alg}_K(A_\alpha, K)$ be the affine variety with coordinate ring A_α . Then there is an obvious injection $X \rightarrow \varprojlim X_\alpha$. Note that the latter is an inverse limit of affine varieties.

Now suppose $X = \varprojlim Y_\gamma$ itself is a provariety over some inverse system $(Y_\gamma, q_{\gamma,\delta}), \gamma \in \mathcal{B}$ and that $A = \Gamma(X, \mathcal{O}_X)$ separates the points of X . Suppose further that the finitely generated K algebras A_α satisfy the following cofinality condition: for $\gamma \in \mathcal{B}$ $\exists \alpha(\gamma) \in \mathcal{A}$ so that $q_\gamma^*(\Gamma(Y_\gamma, \mathcal{O}_{Y_\gamma})) \subseteq A_{\alpha(\gamma)}$. (If each $\Gamma(Y_\gamma, \mathcal{O}_{Y_\gamma})$ is finitely generated, then a cofinal subset of \mathcal{A} always satisfies this.) Then there is a morphism of inverse systems $(Y_\gamma) \rightarrow (X_\alpha)$ and hence a morphism of provarieties $X \rightarrow \varprojlim X_\alpha$. This map is injective under the assumption that A separates the points of X . We record this conclusion, in slightly different notation, in the following proposition:

Proposition 6. *Let $X = \varprojlim X_\alpha$ be a provariety, and assume that each $\Gamma(X_\alpha, \mathcal{O}_{X_\alpha})$ is finitely generated. Suppose that $\Gamma(X, \mathcal{O}_X)$ separates the points of X . Then X embeds in a provariety which is an inverse limit of affine varieties. In particular, X embeds in a product of affine varieties.*

Proof. The only remaining point is the final assertion, which follows from Proposition (5). \square

When A is an (integral domain) K algebra whose K points both separate the elements of A and map surjectively to the K points of its finitely generated subalgebras A_i , the bijection between $\text{Alg}_K(A, K)$ and $\varprojlim \text{Alg}_k(A_i, k)$ identifies the former set theoretically as a provariety X , and, by Corollary (2) (2), identifies A and $\Gamma(X, \mathcal{O}_X)$. For the case of pro-affine algebraic groups, we record this in the following proposition.

Proposition 7. *Let G be a pro-affine algebraic group. Then the provariety structure on G from definition (2) coincides with the structure of G as a pro-algebraic group.*

\square

Finally, we want to deal with pro-affine subgroups of pro-affine algebraic groups.

Theorem 4. *Let H be a pro-affine subgroup of the pro-algebraic group G . Then G/H carries the natural structure of a provariety.*

Proof. We use the set of all finitely generated Hopf subalgebras $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ of $K[G]$ to write the latter as a direct limit and let $\{G_\alpha, p_{\alpha, \beta}\}$ be the corresponding inverse system, so $G = \varprojlim G_\alpha$. We let B_α be the image of A_α in $K[H]$ so that we have an inverse system $\{H_\alpha, q_{\alpha, \beta}\}$ with $H = \varprojlim H_\alpha$. By construction we have injections $H_\alpha \rightarrow G_\alpha$. Set theoretically, we have $G/H = \varprojlim G_\alpha/H_\alpha$. By Proposition (5), we can identify G/H as a provariety as a subset of $\prod G_\alpha/H_\alpha$. \square

For use in our main theorem, we now investigate when the global sections of a provariety G/H as in Theorem (4) separate the points of G/H . We have to allow here for the possibility (in positive characteristic) of bijective non-isomorphisms.

Theorem 5. *Let H be a pro-affine subgroup of the pro-algebraic group G . Then $K[G]^H$ separates the points of G/H if and only if there exists a bijective morphism of provarieties from G/H (with its natural provariety structure) into a provariety $W = \varprojlim W_i$ where each W_i is a quasi-affine algebraic variety.*

Proof. We retain the notation of the proof of Theorem (4). Note that the inverse system $\{G_\alpha/H_\alpha \mid \alpha \in \mathcal{A}\}$ satisfies the transition conditions of Lemma (5). We let K_α denote the kernel of the surjection p_α .

If there is a provariety W as in the theorem, then for a cofinal subset $\mathcal{A}' \subseteq \mathcal{A}$, such that for all $\alpha \in \mathcal{A}'$, there is an injective map $G_\alpha/H_\alpha \subset X_\alpha$ with X_α affine. We can replace X_α by the appropriate affine variety so that $\Gamma(X_\alpha, \mathcal{O}_{X_\alpha}) = K[G_\alpha]^{H_\alpha} = K[G]^{HK_\alpha}$; this makes $G_\alpha/H_\alpha \rightarrow X_\alpha$ a left G equivariant embedding. By Theorem (4) we have an embedding $G/H \rightarrow \prod_{\mathcal{A}'} G_\alpha/H_\alpha$, which we compose with the embedding $\prod_{\mathcal{A}'} G_\alpha/H_\alpha \rightarrow \prod_{\mathcal{A}'} X_\alpha$. It follows that the sections of the pro-affine variety $\prod X_\alpha$ separate the points of G/H , and hence that $\Gamma(G/H, \mathcal{O}_{G/H}) = K[G/H] = K[G]^H$ separates the points of G/H .

Conversely, suppose that $A = K[G]^H$ separates the points of G/H . Let $A_\alpha = K[G]^{HK_\alpha} \subseteq K[G_\alpha]$. A_α is finitely generated over K and left G stable. Let $X_\alpha = \text{Alg}_K(A_\alpha, K)$ be the corresponding affine variety with global sections A_α . G , and G_α , operate on the left on X_α . The function $e : K[G] \rightarrow K$ determines, by restriction, a point $e_\alpha \in X_\alpha$. Let H_α be the stabilizer of e_α in G_α and $H(\alpha)$ the stabilizer of e_α in G , so that $H(\alpha) = p_\alpha^{-1}(H_\alpha)$ and $G_\alpha/H_\alpha = G/H(\alpha)$. Then G_α/H_α embeds in X_α ; we let W_α be its image; W_α is, of course, quasi-affine and the map $G_\alpha/H_\alpha \rightarrow W_\alpha$ is bijective. Note that $H \subseteq H(\alpha)$ for all α . (X_α) , $(G/H(\alpha))$, and (G_α/H_α) form inverse systems, and we have an injective morphism $\varprojlim G_\alpha/H_\alpha \rightarrow \varprojlim X_\alpha$ and an isomorphism $\varprojlim G/H(\alpha) \rightarrow \varprojlim G_\alpha/H_\alpha$. Regard the limits as subsets of the corresponding products. G acts on the products and preserves the inverse limits. We have $e_\infty = (e_\alpha) \in \prod X_\alpha$ belongs to $\varprojlim X_\alpha$. Let H_∞ be the stabilizer of e_∞ , and note that $H_\infty = \cap_{\mathcal{A}} H(\alpha)$, so that $G/H_\infty = \varprojlim G/H(\alpha)$. Also, $H \subseteq H_\infty$ as well.

We have a canonical map $G/H_\infty \rightarrow G/H(\alpha) = G_\alpha/H_\alpha$ for each α , hence a map (necessarily an injection) to $\prod G_\alpha/H_\alpha$ whose image is in $\varprojlim G_\alpha/H_\alpha$. Summarizing, we have maps

$$G/H \rightarrow G/H_\infty \rightarrow \varprojlim G_\alpha/H_\alpha \rightarrow \varprojlim X_\alpha$$

and we consider the composite $\phi : G/H \rightarrow \varprojlim X_\alpha$.

Let $X = \varprojlim X_\alpha$. By construction, $\Gamma(X, \mathcal{O}_X) = A$, and $X = \text{Alg}_K(A, K)$, so that A separates the points of X . Since $\phi^*(\Gamma, \mathcal{O}_X) = A = K[G]^H \subseteq \Gamma(G/H, \mathcal{O}_{G/H})$, and $K[G]^H$ separates the points of G/H , we conclude that ϕ is an injection. (If $\phi(aH) = \phi(bH)$, then $\forall f \in A$, $f(aH) = \phi^*f(aH) = f(\phi(aH)) = f(\phi(bH)) = \phi^*f(bH) = f(bH)$, so $aH = bH$.) It then follows that $G/H \rightarrow G/H_\infty$ is an injection, so that $H = H_\infty$. It follows that

$$G/H = G/H_\infty = \varprojlim G/H(\alpha) = \varprojlim G_\alpha/H_\alpha$$

and

$$\varprojlim G_\alpha/H_\alpha \rightarrow W = \varprojlim W_\alpha$$

so that G/H maps bijectively to W , as desired. \square

With Theorem (5) we have shown the equivalence of conditions (6) and (7) of our Main Theorem. Combined with the equivalences of conditions (1) through (6) shown in Theorem (3), this completes the proof of the Main Theorem.

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