Generic Rings for Picard–Vessiot Extensions and Generic Differential Equations

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Abstract

Let G be an observable subgroup of GL_n. We produce an extension of differential commutative rings generic for Picard–Vessiot extensions with group G.

1 Introduction

Let C be an algebraically closed field of characteristic zero, regarded as a differential field with the trivial derivation, and let G be an algebraic subgroup of $GL_n(C)$. In [3], L. Goldman considered generic differential equations for G, namely an order n (monic, homogeneous, linear) differential equation with the properties that, for differential fields F with constant field C, (1) a Picard–Vessiot extension of F with group G is an extension for a specialization of the equation to F; and (2) for every specialization of the equation to F the corresponding Picard–Vessiot extension has differential Galois group a subgroup of G.

In this paper, we consider the construction of generic ring extensions in a similar context. More precisely, we show that, under the condition of observability of the algebraic subgroup G of $GL_n(C) = GL(V)$, there is a differential ring extension $\mathcal{R} \supseteq \mathcal{R}^G$ which is a generic extension for order n (monic, linear, homogeneous) differential equations (over differential fields F with constants C) with group G in the sense that:

1. If $E \supseteq F$ is a Picard–Vessiot extension for an order n equation with $G(E/F) = G$, then there is a C algebra differential homomorphism $\Phi :$

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$\mathcal{R} \to E$ with $\Phi(\mathcal{R}^G) \subset F$ such that E is the quotient field of $\Phi(\mathcal{R})F$ (first generic property of $\mathcal{R}$); and

2. If $\phi : \mathcal{R}^G \to F$ is any C algebra differential homomorphism and P any maximal differential ideal of $R = F \otimes_\phi \mathcal{R}$ then the quotient field K of R/P is a Picard–Vessiot extension of F with $G(K/F)$ a subgroup of G and the induced homomorphism $\Phi : \mathcal{R} \to K$ has the property that K is the quotient field of $\Phi(\mathcal{R})F$ (second generic property of $\mathcal{R}$).

These results imply corresponding results for generic equations, as we show below (Section 4).

Let $y_1, \ldots, y_n$ be differential indeterminates over C, or more precisely, consider the ring $C\{y_1, \ldots, y_n\}$ of differential polynomials in the y_i's over C. Let $w = \det(y_i^{(j)})$ be the Wronskian determinant of the y_i's and consider, finally, the differential C algebra $\mathcal{R} = C\{y_1, \ldots, y_n\}[w^{-1}]$. As we recall below, there are elements $b_0, \ldots, b_{n-1}$ of $\mathcal{R}$ such that the y_i's all satisfy the linear homogeneous differential equation $Y^{(n)} + b_{n-1}Y^{(n-1)} + \cdots + b_0 Y = 0$. Moreover, the b_i's are all differentially independent over C, so that $\mathcal{R}$ is an algebra over its subring $C\{b_0, \ldots, b_n\}$.

Now suppose that F is a differential field with field of constants C, that $Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_0 Y = 0$ is a differential equation over F and that $E \supseteq F$ is a Picard–Vessiot (differential Galois) extension for this equation. Let $u_1, \ldots, u_n$ in E be a (full) set of solutions, linearly independent over C (hence with non–zero Wronskian.) Because the y_i are differential indeterminates, there is a differential C algebra homomorphism $C\{y_1, \ldots, y_n\} \to E$ with $y_i \mapsto u_i$, and since the Wronskian of the u_i is non–zero this extends to a differential C algebra homomorphism $\Phi : \mathcal{R} \to E$. It follows from the definitions that E is the quotient field of the subring $F\Phi(\mathcal{R})$, and it is elementary that $\Phi(b_i) = a_i$.

On the other hand, if $d_0, \ldots, d_{n-1}$ are any elements of F and $\phi : C\{b_0, \ldots, b_{n-1}\} \to F$ is the differential C algebra homomorphism with $\phi(b_i) = d_i$, we can consider the F algebra $R = F \otimes_\phi \mathcal{R}$. Because the y_i's satisfy the differential equation with coefficients b_i, their images $z_i = 1 \otimes y_i$ in R satisfy the differential equation with coefficients $\phi(b_i) = d_i$. In particular, R is finitely generated as an (ordinary) F algebra. If we select a maximal differential ideal P of R, mod it out, and take the quotient field K of the ring R/P, we then have that K is a Picard–Vessiot extension of F for $Y^{(n)} + d_{n-1}Y^{(n-1)} + \cdots + d_0 Y = 0$, and the composite $\mathcal{R} \to R \to R/P \to K$, which we denote by Φ, has the property that K is the quotient field of $F\Phi(\mathcal{R})$ and that $\Phi(b_i) = d_i$.

Because of the above, one could say in some weak sense that the equation $Y^{(n)} + b_{n-1}Y^{(n-1)} + \cdots + b_0 Y = 0$ is generic for order n (monic, linear, homogeneous) differential equations over differential fields with field of constants C, and that $\mathcal{R}$ is the ring of generic solutions of the generic equation. In the same sense, $\mathcal{R} \supset C\{b_0, \ldots, b_n\}$ is generic for order n Picard–Vessiot extensions.

The discussion so far, however, has ignored differential Galois groups. To bring them into the picture, we consider the n-dimensional C vector space

$V = Cy_1 + \cdots + Cy_n$. In the above notation(s), $\Phi : V \to E$ (or $\Phi : V \to K$) is an injection, and the differential Galois group, in either case, is represented faithfully on $\Phi(V)$ and hence in $GL(\Phi(V))$. Additionally, *every* linear automorphism of V extends to a differential automorphism of $\mathcal{R}$ which leaves $C\{b_0, \ldots, b_n\}$ fixed. But these differential automorphisms need not commute with Φ in either case. Thus in the above characterization of $\mathcal{R} \supset C\{b_0, \ldots, b_n\}$ as a generic extension, there is no group G.

We retain the notation C introduced above. For terminology and basic results regarding differential Galois extensions with linear algebraic differential Galois group (Picard–Vessiot extensions) we refer to [5] and [6]. If $E \supset F$ is a differential field extension, we will denote the derivation D_E by D when no ambiguity arises. If S is a subset of E, we let $F\langle S \rangle$ denote the smallest differential subfield of E that contains both F and S. If S is a subset of the differential ring T and R is a differential subring of T we let $R\{S\}$ denote the smallest differential subring of T that contains both R and S. We denote the field of constants of E by C_E. The extension has *no new constants* if $C_E = C$. For an element y of any extension E, we use y' and $y^{(n)}$ to denote $D(y)$ and $D^n(y)$ as usual. We always use $G(E/F)$ to denote the group of differential automorphisms of E over F. We begin by recalling some notation, definitions, and standard results:

If $E \supset F$ is a *Picard–Vessiot*, or *Differential Galois*, extension for an order n monic linear homogeneous differential operator

$$L = Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_1 Y^{(1)} + a_0 Y; \qquad a_i \in F \qquad (1)$$

and $V = \{y \in E \mid L(y) = 0\}$ then E is differentially generated over F by V, the constants of E are those of F ("no new constants"), and $\dim_C(V) = n$ ("full set of solutions").

For Picard–Vessiot extensions, $G(E/F) \to GL(L^{-1}(0))$ is an injection with Zariski closed image.

We retain the conventions and the notations (and the choices made in introducing those notations) of this introduction throughout.

2   The Ring of Generic Solutions

We begin by considering the ring $C\{y_1, \ldots, y_n\}$ of differential polynomials in the differential indeterminates $y_1, \ldots, y_n$ over the constant field C. We let the group $GL_n(C)$ act in the standard way on the n-dimensional C vector space $Cy_1 + \cdots + Cy_n$; this action extends linearly to a rational action by differential automorphisms on $C\{y_1, \ldots, y_n\}$ (see [5, Example 3.29, p.37]). We introduce the following notation

Notation 1. *Let $\mathcal{W}$ denote the $n + 1 \times n$ matrix*

$$\begin{bmatrix} y_1^{(0)} & y_1^{(1)} & \cdots & y_1^{(n)} \\ y_2^{(0)} & y_2^{(1)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(0)} & y_n^{(1)} & \cdots & y_n^{(n)} \end{bmatrix}$$

Let $\mathcal{W}_i$, $0 \leq i \leq n$ denote the $n \times n$ matrix obtained from $\mathcal{W}$ by deleting column i from it. Let w_i denote the determinant of $\mathcal{W}_i$.

Now let Y be an additional differential indeterminate over C. The w_i's, up to sign, are the coefficients of the Wronskian determinant $w(Y, y_1, \ldots, y_n)$

$$\begin{vmatrix} Y^{(0)} & y_1^{(0)} & \cdots & y_n^{(0)} \\ Y^{(1)} & y_1^{(1)} & \cdots & y_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{(n)} & y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

namely $w(Y, y_1, \ldots, y_n) = \sum_{i=0}^{n} (-1)^i w_i Y^{(i)}$.

Under the $GL_n(C)$ action on $C\{y_1, \ldots, y_n\}$ each w_i is a semi-invariant of weight det [5, Proposition 2.6, p.17]. In particular, this means that the $GL_n(C)$ action extends to a rational action on the differential ring $C\{y_1, \ldots, y_n\}[w_n^{-1}]$. This latter is our ring of generic solutions.

Definition 1. $\mathcal{R} = C\{y_1, \ldots, y_n\}[w_n^{-1}]$ *is called the* ring of generic solutions *of a linear monic order n equation. We regard the previously specified $GL_n(C)$ action on $\mathcal{R}$ as part of this definition.*

Notation 2. *For $0 \leq i \leq n-1$, let $b_i \in \mathcal{R}$ denote $(-1)^i w_i w_n^{-1}$ so that*

$$w_n^{-1} w(Y, y_1, \ldots, y_n) = Y^{(n)} + b_{n-1} Y^{(n-1)} + \cdots + b_0 Y^{(0)}.$$

We let $\mathcal{L}(y_1, \ldots, y_n)(Y)$ denote the differential operator on the right hand side of this equation. When no confusion arises, we will simply denote this operator $\mathcal{L}(Y)$.

We note that each b_i is a $GL_n(C)$ invariant of $\mathcal{R}$ and that each y_i satisfies $\mathcal{L}(y_i) = 0$. Further, the b_i's are differentially independent over C [5, Theorem 2.17, p.22] so that $C\{b_0, \ldots, b_{n-1}\}$ is a differential C subalgebra of $\mathcal{R}$. Using $\mathcal{L}$, the $y_i^{(j)}$ for $j \geq n$ can be expressed as linear combinations of $y_i^{(0)}, \ldots, y_i^{(n-1)}$, from which it follows that

$$\mathcal{R} = C\{b_0, \ldots, b_{n-1}\}[y_i^{(j)}, 1 \leq i \leq n, 0 \leq j \leq n-1][w_n^{-1}]. \qquad (2)$$

This is a localization of an (ordinary) polynomial ring extension: the $y_i^{(j)}$ are algebraically independent over $C\{b_0, \ldots, b_{n-1}\}$, by [5, Theorem 2.17, p.22].

Now we make explicit the structure of $\mathcal{R}$ as a $GL_n(C)$ module.

To begin this task, we note that $\mathcal{R}$ can be regarded as the coordinate ring of the (infinite) algebraic variety $GL_n(C) \times V$, where V denotes a countable

product of the space $Cy_1 + \cdots + Cy_n$. In this identification, $\{y_i^{(j)} \mid 1 \leq i \leq n, 0 \leq j \leq n-1\}$ are coordinates on $GL_n(C)$ ($y_i^{(j)}$ gives the entry in row i and column $j+1$ of a matrix in $GL_n(C)$), w_n is the determinant on $GL_n(C)$, and for each $j \geq n$, $y_1^{(j)}, \ldots y_n^{(j)}$ are the coordinates of a copy of $Cy_1 + \cdots + Cy_n$. Let V_t denote the same space V except with trivial $GL_n(C)$ action, and let $I : V \to V_t$ be the identity function. There is a standard $GL_n(C)$ equivariant bijection $GL_n(C) \times V \to GL_n(C) \times V_t$ given by $(g,v) \mapsto (g, I(g^{-1}v))$ (with inverse $(g,v) \to (g, gI^{-1}(v))$). Thus $\mathcal{R}$ becomes, under this isomorphism, the coordinate ring of $GL_n(C) \times V_t$ so that $\mathcal{R} \cong C[GL_n] \otimes C[V_t]$ as C algebras with $GL_n(C)$ action. In this identification $C[GL_n]$ is $C[y_i^{(j)}, 1 \leq i \leq n, 0 \leq j \leq n-1][w_n^{-1}]$ and $C[V_t]$ is $\mathcal{R}^{GL_n}$. This implies that

$$\mathcal{R} = \mathcal{R}^{GL_n(C)} \otimes_C C[y_i^{(j)}, 1 \leq i \leq n, 0 \leq j \leq n-1][w_n^{-1}]. \tag{3}$$

The b_i's are $GL_n(C)$ invariants, and hence $C\{b_0, \ldots, b_{n-1}\}$ is contained in $\mathcal{R}^{GL_n(C)}$. Combining (2) and (3), we conclude:

Proposition 1. *For $\mathcal{R} = C\{y_1, \ldots, y_n\}$ with $GL_n(C)$ action extended from the linear action on $V = Cy_1 + \cdots + Cy_n$ using the basis $y_1, \ldots, y_n$, we have*

1. $\mathcal{R}^{GL_n(C)} = C\{b_0, \ldots, b_{n-1}\};$

2. As algebras with $GL_n(C)$ action,

$$\mathcal{R} = \mathcal{R}^{GL_n(C)} \otimes_C C[y_i^{(j)}, 1 \leq i \leq n, 0 \leq j \leq n-1][w_n^{-1}]$$
$$= \mathcal{R}^{GL_n(C)} \otimes_C C[GL_n(C)].$$

It is important to remember that the tensor product decomposition in Proposition (1) is *not* as differential algebras. Indeed, the differential subalgebra generated by the second factor $C[GL_n(C)]$ is all of $\mathcal{R}$.

3 Order n Extensions

Definition 2. *A Picard–Vessiot extension $E \supset F$ is said to be of* order n *if it is a Picard–Vessiot extension for some monic linear differential operator over F of order n.*

The following result is well known. We include it here for completeness and the lack of convenient reference.

Lemma 1. *Assume that F contains a non-constant. Let $E \supset F$ be a Picard–Vessiot extension with $G(E/F)$ infinite. Let W be any faithful finite-dimensional rational module for $G(E/F)$. Then there is a $G(E/F)$ module injection $\psi : W \to E$ and $E = F\langle \psi(W) \rangle$.*

Proof. Let G denote $G(E/F)$. Let V be any rational G submodule of E on which G acts faithfully, and let $K = F\langle V \rangle$. By the Fundamental Theorem of differential Galois theory, $K = E^H$, where H is the subgroup of G consisting of elements which are the identity on K. Thus H acts trivially on V, and hence H is trivial, and so $K = E$. Thus it suffices to prove the existence of the injection ψ.

We will use a few properties of rational, not necessarily finite dimensional, G modules, namely that that the coordinate ring $C[G]$ is a rationally injective G module and that the tensor product of a rationally injective G module and an arbitrary rational G module is rationally injective [1, p.4]. We also observe that any rational G module, in particular a finite diemnsional one, is an essential extension of its socle.

Let T denote the sum of all the rational G submodules of E. We recall that by Kolchin's Theorem [5, Theorem 5.12 p.67] there is a G module and algebra isomorphism

$$\overline{F} \otimes_F T \cong \overline{F} \otimes_C C[G].$$

(Here $\overline{F}$ is the algebraic closure of F.) In fact, the isomorphism already occurs for a finite extension $K \supset F$ in place of $\overline{F}$. Since K is a trivial G module, and by assumption is infinite dimensional over C, this isomorphism can be written

$$\oplus T \cong \oplus C[G]$$

the number of summands on the left being equal to $[K : F]$ and that on the right being infinite. We note that the right hand side is an injective G module (since $C[G]$ is an injective G module and the direct sum can be considered as the tensor product of $C[G]$ and an infinite dimensional trivial module), and hence so is the left, and hence so is T, and that each simple finite dimensional G module occurs with infinite multiplicity in the right hand side, hence on the left, and hence also in T. Because T contains representatives of all simple G modules, each with at least countable mutiplicity, it follows that T contains a copy of every finite dimensional semi-simple module. Now suppose that V is a finite dimensional G module with socle V_s. Since this socle is semi-simple, there is a embedding $V_s \to T$ which, since T is injective, extends to a G morphism $V \to T$. The kernel of this morphism has trivial socle, and hence is trivial, so in fact W embeds in T. We conclude that every finite dimensional G module occurs in T. $\qquad\square$

Lemma 1 immediately implies the first universal property of the ring $\mathcal{R}$:

Theorem 1. *Assume that F contains a non-constant, that $E \supset F$ is a Picard–Vessiot extension with Galois group $G(E/F)$, and that $G(E/F)$ has a faithful representation ρ in $GL_n(C)$. Then $E \supset F$ is of order n and there is a differential homomorphism $\Psi : \mathcal{R} \to E$ such that*

1. E is the quotient field of $F\Psi(\mathcal{R})$; and

2. E is a Picard–Vessiot extension of F for

$$L = Y^{(n)} + \Psi(b_{n-1})Y^{(n-1)} + \cdots + \Psi(b_0)Y^{(0)}; \text{ and}$$

3. Ψ is $G(E/F)$ equivariant, so $\Psi(\mathcal{R}^{G(E/F)})$ is contained in $E^{G(E/F)} = F$.

Proof. Let $G = G(E/F)$. Let W denote the G module obtained by G acting on $\sum Cy_i$ via ρ. By Lemma 1, we have an embedding $\psi : W \to E$, and we know that the image of this embedding generates E over F differentially. Note that the elements $\psi(y_i)$ are linearly independent over C, and thus have non–zero Wronskian. Hence we can define a differential homomorphism $\Psi : \mathcal{R} \to E$ which sends y_i to $\psi(y_i)$. Let $W^{(j)} = \sum Cy_i^{(j)}$ and let $\psi^{(j)}$ be the restriction of Ψ to $W^{(j)}$. Then $W^{(0)} = W$ and $\psi^{(0)} = \psi$ and therefore is G equivariant, and it follows that each $\psi^{(j)}$ is G equivariant as well. Then the canonical extension of each $\psi^{(j)}$ to a map from the symmetric algebra $S_C(W^{(j)})$ to E is G equivariant, and so is the tensor product of all of these. But this is the map $C\{y_1, \ldots, y_n\} \to E$ used to produce Ψ. It then follows that Ψ has the specified properties. $\square$

In Theorem 1 we showed that $\Psi(\mathcal{R}^{G(E/F)})$ is contained in $E^{G(E/F)} = F$. We now turn our attention to homomorphisms with this property. More precisely, we now fix a subgroup G of $GL_n(C)$, a differential field F with field of constants C, and we consider differential homomorphisms $\phi : \mathcal{R}^G \to F$.

Let α denote the restriction of ϕ to $\mathcal{R}^{GL_n(C)}$ and consider the differential F algebra $R = F \otimes_\alpha \mathcal{R}$. Note that $R = F[y_1^0, \ldots, y_n^{(n-1)}][w_n^{-1}]$. We can factor ϕ through α, and we denote the corresponding homomorphism $R^G \to F$ by ϕ as well. Let Q denote the kernel of ϕ, and let P be any maximal differential ideal of R lying over Q. Now we take the quotient field E of R/P. Let $\Phi : \mathcal{R} \to E$ be the resulting homomorphism. Then Φ extends ϕ. By [5, Corollary 1.18, p. 11], we know that E has field of constants C. We also observe that E is differentially generated over F by the $\Phi(y_i)$, which are linearly independent over C since their Wronskian, $\Phi(w_n)$, is necessarily non-zero, and that each $\Phi(y_i)$ is a solution of the monic linear differential equation $L = Y^{(n)} + \phi(b_{n-1})Y^{(n-1)} + \cdots + \phi(b_0)Y^{(0)}$. It follows that E is a Picard–Vessiot extension of F. We now show that, provided that G is observable in $GL_n(C)$ [4], that $G(E/F)$ must be a subgroup of G (second generic property of $\mathcal{R}$).

Theorem 2. *Assume that G is an observable subgroup of $GL_n(C)$. Let $\phi : \mathcal{R}^G \to F$ be a differential F algebra homomorphism, let α be the restriction of ϕ to $\mathcal{R}^{GL_n(C)}$. Let P be a maximal differential ideal of $R = F \otimes_\alpha \mathcal{R}$ whose inverse image in $\mathcal{R}$ contains the kernel of ϕ, and let E be the fraction field of R/P. Then E is a Picard–Vessiot extension of F with $G(E/F)$ a subgroup of G.*

Proof. All the assertions of the theorem have already been established, except the final one, in the discussion immediately preceding it. We resume that discussion, keeping the same notation. By Proposition 1, $\mathcal{R} = \mathcal{R}^{GL_n(C)} \otimes C[GL_n(C)]$

and hence $R = F \otimes_\alpha \mathcal{R} = F \otimes C[GL_n(C)]$. Since F is a flat, indeed free, C module, we have $R^G = (F \otimes C[GL_n(C)])^G = F \otimes (C[GL_n(C)]^G)$. Let S be the subring of E consisting of all elements satisfying a linear differential equation over F. By construction, R/P is a subring of S. We have the commutative diagram

$$
\begin{array}{ccc}
R & \longrightarrow & S \\
\uparrow & & \uparrow \\
R^G & \longrightarrow & F
\end{array}
$$

which can also be written

$$
\begin{array}{ccc}
F[GL_n(C)] & \longrightarrow & S \\
\uparrow & & \uparrow \\
F[GL_n(C)]^G & \longrightarrow & F
\end{array}
$$

The action of $GL_n(C)$ on $R = F[GL_n(C)]$ is the standard one, and the map $\mathcal{R} \to R$ is $GL_n(C)$ equivariant. The map $R \to S$ is not $GL_n(C)$ equivariant; in general, $GL_n(C)$ does not act on S. However, the stabilizer $GL_n(C)_P$ in $GL_n(C)$ of the kernel P acts as differential automorphisms on R/P, its quotient field E, and hence the ring S. In [5, Theorem 4.14, p.49], it is shown that this action gives an isomorphism of $GL_n(C)_P$ and $G(E/F)$. In particular, we have that $R \to S$ is $G(E/F)$ equivariant.

Now we apply Kolchin's Theorem (*op. cit.*) again: tensor over F with the algebraic closure $\overline{F}$ to obtain

$$
\begin{array}{ccc}
\overline{F}[GL_n] & \longrightarrow & \overline{F} \otimes_F S \\
\uparrow & & \uparrow \\
\overline{F}[GL_n]^G & \longrightarrow & \overline{F}
\end{array}
$$

Because G is observable in $GL_n(C)$, we have that $\overline{F}[GL_n]^G$ is the coordinate ring of an affine variety densely containing the coset variety $G\backslash GL_n$. And by Kolchin's Theorem, we have that $\overline{F} \otimes_F S$ is the coordinate ring of $G(E/F)$ with scalars extended to $\overline{F}$. Hence the above diagram of rings corresponds to the diagram

$$
\begin{array}{ccc}
GL_n & \longleftarrow & G(E/F) \\
\downarrow & & \downarrow \\
G\backslash GL_n & \longleftarrow & (\text{pt})
\end{array}
$$

Since the top horizontal map is $G(E/F)$ equivariant, this diagram implies that a coset of $G(E/F)$ is contained in a coset of G, and hence that $G(E/F)$ is a subgroup of G. $\qquad\square$

Remark 1. *In the situation and notation of Theorem 2, there may not be any ideals P. For convenience, we assume that $F = \overline{F}$ and let X be the affine variety*

with coordinate ring $\overline{F}[GL_n]^G$. Then Q corresponds to a point of X, and if it happens to lie in $X - G\backslash GL_n$ there will be no ideal of R over Q. In case G is not simply observable, but is actually co–affine (meaning that $G\backslash GL_n$ is affine), then P's always exist for any Q. Reductive subgroups are always co–affine, and since GL_n is reductive and the characteristic is zero, these are the only ones.

We exhibit an example where the situation of Remark 1 occurs below (Example 1). Here, we record the existence consequences of Remark 1 as a corollary to Theorem 2.

Corollary 1. *Assume that G is a reductive subgroup of $GL_n(C)$. Let $\phi : \mathcal{R}^G \to F$ be a differential F algebra homomorphism, and let α be the restriction of ϕ to $\mathcal{R}^{GL_n(C)}$. Then there exists a maximal differential ideal P of $R = F \otimes_\alpha \mathcal{R}$ whose inverse image in $\mathcal{R}$ contains the kernel of ϕ. The fraction field E of R/P is a Picard–Vessiot extension of F with $G(E/F)$ a subgroup of G.*

Example 1. *Let $n = 2$, $F = C(x)$ (rational functions with $x' = 1$) and $G = G_a \leq GL_n(C)$. To simplify notation, we will write $C[GL_2]$ as $C[a, b, c, d][(ad - bc)^{-1}]$ and let G_a with coordinate t act by $b \mapsto b + ta$, $d \mapsto d + tc$, $a \mapsto a$, and $c \mapsto c$. Then $C[GL_2]^G = C[a, c, (ad - bc), (ad - bc)^{-1}]$. Then we think of $\mathcal{R}$ as $C\{a, b\}[w^{-1}]$ with $c = a'$, $d = b'$, and $w = ad - bc$. Choose ϕ so that $\phi(b_0) = \phi(b_1) = 0$ (so the differential equations is $Y'' = 0$). Then in $F[GL_2]$ we have $a'' = b'' = 0$. Then define ϕ on $F[GL_2]^G$ to satisfy $\phi(a) = \phi(c) = 0$ and $\phi(ad - bc) = 1$, which is easily checked to be a differential homomorphism. But there is no extension of ϕ to a homomorphism, differential or not, $F[GL_2] \to T$ for any ring $T \supseteq F$: both a and c would be sent to zero, which precludes sending $ad - bc$ to a non–unit. Explicitly, Q here is the ideal generated by a, c, and $(ad - bc) - 1$, which is proper in $F[GL_2]^G$ but not in $F[GL_2]$. Geometrically, $G_a\backslash GL_2$ is embedded in $C \times C \times (C - \{0\})$ via*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, c, ad - bc).$$

4 Generic Equations

As noted in the introduction, our results touch on previous investigations of Goldman [3] (see also [2]) and in fact were inspired by trying to transform that work from the language of fields and specializations to the language of rings and homomorphisms. Goldman works with equations, and to explain the connections we will need some additional comments on our results. Goldman's groups are all reductive, and we make that simplifying assumption as well.

We fix a reductive algebraic subgroup G of $GL_n(C)$ and consider a homomorphism $\phi : \mathcal{R}^G \to F$ as in Theorem 2 as well as the rings R and S, and the field E, in the notation of the Theorem and its proof. (We also denote the homomorphism $R^G \to F$ by ϕ, as in the theorem.) The algebraic structure of R is independent of ϕ: it is simply the coordinate ring of GL_n. The differential structure of R is completely determined by the restriction α of ϕ

to $\mathcal{R}^{GL_n(C)} = C\{b_o, \ldots, b_{n-1}\}$, that is it is determined by the elements $\phi(b_i)$, $0 \leq i \leq n-1$, of F. The maximal differential ideals P of R are all conjugate under the $GL_n(C)$ action [5, Theorem 4.17 p.50], which means that the quotient R/P and its quotient field E are also determined by the $\phi(b_i)$'s. These observations are making explicit the fact that a Picard–Vessiot extension like E for $Y^{(n)} + \phi(b_{n-1})Y^{(n-1)} + \cdots + \phi(b_1)Y^{(1)} + \phi(b_0)Y = 0$ is determined up to isomorphism.

We can rephrase our main genericity results in terms of this equation as follows:

1. If $E \supseteq F$ is a Picard–Vessiot extension for an order n equation with $G(E/F) = G$, then E is a Picard–Vessiot extension for $Y^{(n)} + \Phi(b_{n-1})Y^{(n-1)} + \cdots + \Phi(b_1)Y^{(1)} + \Phi(b_0)Y = 0$, where $\Phi : \mathcal{R} \to E$ is a C algebra differential homomorphism with $\Phi(\mathcal{R}^G) \subset F$, such that E is the quotient field of $\Phi(\mathcal{R})F$ (first generic property of $\mathcal{R}$); and

2. If $\phi : \mathcal{R}^G \to F$ is any C algebra differential homomorphism and K is a Picard–Vessiot extension of F for $Y^{(n)} + \phi(b_{n-1})Y^{(n-1)} + \cdots + \phi(b_1)Y^{(1)} + \phi(b_0)Y = 0$ then $G(K/F)$ is a subgroup of G and the induced homomorphism $\Phi : \mathcal{R} \to K$ has the property that K is the quotient field of $\Phi(\mathcal{R})F$ (second generic property of $\mathcal{R}$).

Thus one might consider $Y^{(n)} + b_{n-1}Y^{(n-1)} + \cdots + b_1Y^{(1)} + b_0Y = 0$ as a generic equation for G, whose specializations yield the Picard–Vessiot extensions with group G. However, as we have seen, the allowable "specializations" $\phi(b_i)$ of the b_i are those that come from differential homomorphisms $\phi : \mathcal{R}^G \to F$. To describe these, suppose that t_i, $1 \leq i \leq m$ are such that $\mathcal{R}^G = C\{t_1, \ldots, t_m\}$. We are not assuming that the t_i are differentially independent; let $\mathcal{S}$ be a generating set for their differential relations. Then differential homomorphisms ϕ on $\mathcal{R}^G$ are specified by the m-tuple, $(\phi(t_1), \ldots, \phi(t_m))$, and any m-tuple of elements satisfying all the relations in $\mathcal{S}$ produces such a homomorphism. Since $b_i \in \mathcal{R}^G$, $0 \leq i \leq n-1$, we can express each b_i as a differential polynomial in the t_i, say $b_i = f_i(t_1, \ldots, t_m)$. With these notations, we have the following reformulation in the language of equations of our main results:

Theorem 3. *Let G be a reductive subgroup of $GL_n(C)$. Suppose that $\mathcal{R}^G$ is generated by $t_1, \ldots, t_m$, subject to the relations $\mathcal{S}$. Let*

$$L_{t_1, \ldots, t_m} = Y^{(n)} + \sum_0^{n-1} f_i(t_1, \ldots, t_m)Y^{(i)}.$$

Then if $E \supseteq F$ is a Picard–Vessiot extension of order n with group G, there are elements a_i, $1 \leq i \leq m$ satisfying the relations $\mathcal{S}$ such that E is a Picard–Vessiot extension of F for $L_{a_1, \ldots, a_m}$. Conversely, if a_i, $1 \leq i \leq m$ is a set of elements of F satisfying the relations $\mathcal{S}$ and $K \supseteq F$ is a Picard–Vessiot extension for $L_{a_1, \ldots, a_m}$ then $G(K/F)$ is a subgroup of G.

Note: because G is reductive, $C[GL_n]^G$ is finitely generated as an algebra over C. Since $\mathcal{R}^G = C\{b_0, \ldots, b_{n-1}\} \otimes C[GL_n]^G$, we have a finite set of differential generators $t_1, \ldots, t_m$. For the set $\mathcal{S}$, we can take a (set of generators for) the kernel of a surjection $C\{T_1, \ldots, T_m\} \to \mathcal{R}^G$ where the T_i are differential indeterminates and $T_i \mapsto t_i$. In the cases considered by [2] and [3], the set $\mathcal{S}$ is essentially empty: those authors use elements t_i of $C\langle y_1, \ldots, y_n \rangle$ such that $C\langle y_1, \ldots, y_n \rangle^G = C\langle t_1, \ldots, t_n \rangle$ which are differentiably independent. The t_i's are quotients of elements of $\mathcal{R}^G$ and for ϕ to be defined on them some conditions apply.

Example 2. *As an example of Theorem 3, we may consider the case that $G = SL_n$. Then $C[GL_n]^G = C[det, det^{-1}]$ so that $\mathcal{R}^G = C\{b_0, \ldots, b_{n-1}, w_n, w_n^{-1}\}$ But since $b_i = (-1)^i w_i w_n^{-1}$, this means that $\mathcal{R}^G$ is differentially generated over C by $w_0, \ldots, w_n$. We can consider $w_0, \ldots, w_{n-1}$ as differential indeterminates. For the Wronskian w_n, we have the derivative formula $w_n' = -b_{n-1} w_n$ [5, 2.4.2, p. 16]; since $b_{n-1} = (-1)^{n-1} w_{n-1} w_n^{-1}$ this means that $w_n' = (-1)^n w_{n-1}$. Thus the set $\mathcal{S}$ consists of the single relation $w_n' = (-1)^n w_{n-1}$, plus the condition that $w_n \neq 0$. Or, to revert the notation of Theorem 3, with $t_i = w_{i-1}$, $1 \leq i \leq n+1$, and $f_{i-1} = \frac{t_i}{t_{n+1}}$, if $E \supset F$ is an order n Picard–Vessiot extension with group $SL_n(C)$, then there are elements $a_1, \ldots, a_{n+1}$ of F with $a_{n+1} \neq 0$ and $a_{n+1}' = a_n$ such that E is a Picard–Vesiot extension for $L_{a_1, \ldots, a_n}$. Conversely, given any such elements of F, the Picard–Vessiot extension E of F for $L_{a_1, \ldots, a_n}$ has $G(E/F)$ a subgroup of $SL_n(C)$.*

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