Number Theory Fall 2009
Homework 13
Due: Wed. Dec. 9, start of class

Throughout, K denotes a number field.

11.4 PID's 'n ED's

Exercise 11.7. *Suppose R is a Euclidean domain with absolute value $|\cdot|$. Let $a \in R$ such that $|a| = 1$. Show $(a) = R$.*

11.6 Quotient rings

Exercise 11.8. *Let $R = \mathbb{Z}[\sqrt{-5}]$ and consider the ideals $\mathcal{I} = (2)$, $\mathcal{J} = (1+\sqrt{-5})$ and $\mathfrak{p} = (2, 1+\sqrt{-5})$. Write down a set of representatives for $R/\mathcal{I}$, $R/\mathcal{J}$ and $R/\mathfrak{p}$. What are their norms? Show that $\mathcal{I}$ and $\mathcal{J}$ are not prime but $\mathfrak{p}$ is.*

12.2 Fractional ideals and the class group

Definition 12.1. *Let K be a number field, and $\mathcal{I} \subseteq K$. If $a\mathcal{I} = \{ai : i \in \mathcal{I}\}$ is an ideal of $\mathcal{O}_K$ for some nonzero $a \in \mathcal{O}_K$, we say $\mathcal{I}$ is a* **fractional ideal** *of $\mathcal{O}_K$. Further if $a\mathcal{I}$ is principal, we say $\mathcal{I}$ is* **principal**. *Denote the set of nonzero fractional ideals of $\mathcal{O}_K$ by $\mathrm{Frac}(\mathcal{O}_K)$, and the set of nonzero principal ideals by $\mathrm{Prin}(\mathcal{O}_K)$.*

Exercise 12.1. *Let $\mathcal{I} = (n)$ be a non-zero ideal of $\mathbb{Z}$. Check the fractional ideal $\mathcal{I}^{-1} = \frac{1}{n}\mathbb{Z}$ is indeed the inverse of $\mathcal{I}$, i.e., $\mathcal{I}\mathcal{I}^{-1} = (1) = \mathbb{Z}$. Similarly, for any number field K and any non-zero principal ideal $\mathcal{I} = (\alpha)$ of $\mathcal{O}_K$, show $\alpha^{-1}\mathcal{O}_K$ is the inverse of $\mathcal{I}$, i.e., $\mathcal{I}\mathcal{I}^{-1} = (1) = \mathcal{O}_K$.*

Exercise 12.2. *Let K be a number field. Show the principal fractional ideals of $\mathcal{O}_K$ correspond to the elements of K, up to units.*

Exercise 12.3. *Check that $\mathcal{O}_K$ is the identity element of $\mathrm{Frac}(\mathcal{O}_K)$, i.e., if $\mathcal{I}$ is a fractional ideal of $\mathcal{O}_K$, show $\mathcal{O}_K \cdot \mathcal{I} = \mathcal{I}$.*

Exercise 12.4. *Let $\mathcal{I}, \mathcal{J}$ be ideals of $\mathcal{O}_K$. Then $\mathcal{J}|\mathcal{I} \iff \mathcal{J} \supseteq \mathcal{I} \iff \mathcal{J}^{-1} \subseteq \mathcal{I}^{-1}$. (Hint: it's easy if you use Theorem 12.4 to multiply by inverses.) Note when $\mathcal{J} = \mathcal{O}_K$, this says $\mathcal{I}^{-1} \supseteq \mathcal{O}_K$.*

12.3 Primes of the form $x^2 + 5y^2$

Exercise 12.5. *Use Fermat's 2 square theorem to determine the primes of the form $x^2 + 4y^2$ (Exercise 12.8.1).*