Calculus III (2934, Fall 2019)
Worksheets

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Worksheet 1: $\mathbb{R}^2$ and $\mathbb{R}^3$

1. What does $\mathbb{R}^2$ mean? Can you give a precise definition? What about $\mathbb{R}^3$? $\mathbb{R}^n$?
2. What is the right-hand rule (la regolla della mano destra)?
3. Graph $x = y$ in $\mathbb{R}^2$.
4. Graph $x = y$ in $\mathbb{R}^3$.
5. Graph $x^2 + y^2 = 1$ in $\mathbb{R}^3$.
6. Geometrically describe $\left\{(x, y, z) : 1 \leq \sqrt{x^2 + y^2 + z^2} \leq 4\right\}$.
7. Write an equation for the sphere of radius 5 in $\mathbb{R}^3$ centered at $(1, 2, 3)$.
8. Find the distance between the points $(1, 2, 3)$ and $(2, 4, 6)$ in $\mathbb{R}^3$.
9. Graph $x + y + z = 1$.
10. Graph $x^2 + y^2 = z$ in $\mathbb{R}^3$.
11. Graph $x^2 + y^2 = z^2$ in $\mathbb{R}^3$.
12. What are your strategies for graphing equations/inequalities in $\mathbb{R}^3$?

Worksheet 2: Vectors

1. What is a vector in $\mathbb{R}^2$? $\mathbb{R}^3$? What is the difference between a vector and a point? How do you denote them algebraically?

2. What does the sum of 2 vectors mean? (algebraically and geometrically) Is there a difference for $\mathbb{R}^2$ and $\mathbb{R}^3$?

3. What does the product of 2 vectors mean? (algebraically and geometrically) Is there a difference for $\mathbb{R}^2$ and $\mathbb{R}^3$?

4. Can one similarly add and multiply points?

5. What is the determinant of a 2×2 matrix? What does it mean geometrically?

6. What is the determinant of a 3×3 matrix?

7. What algebraic properties do sums and products of vectors satisfy?

Worksheet 3: Products of Vectors

1. What is the difference between scalar, dot and cross products? When do they make sense? What do they mean geometrically?

2. Do all of these products commute? E.g., is $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ always?

3. Do any of these products preserve length? E.g., is $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}|$?

4. Do any of these products have a cancellation property? E.g., does $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ imply $\mathbf{v} = \mathbf{w}$? (maybe assuming something is nonzero)

5. Use the dot product to compute the projections of $\mathbf{u} = \langle 2, 1 \rangle$ onto $\mathbf{v} = \langle 1, 0 \rangle$ and onto $\mathbf{w} = \langle 1, -1 \rangle$. Can you interpret the projections in terms of work?

6. Is there a difference between the projection of a $\mathbf{u}$ onto $\mathbf{v}$ and the projection of $\mathbf{u}$ onto $-\mathbf{v}$? What about the projection of $\mathbf{v}$ onto $\mathbf{u}$?

7. Compute $\mathbf{i} \times \mathbf{j}$ and $\langle 1, 2, 3 \rangle \times \langle 1, 0, -1 \rangle$. Sketch pictures.

8. Consider vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$. Explain why $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ being coplanar means the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$. Is the converse also true?

Worksheet 4: Lines and Planes

1. Find vector, parametric and symmetric equations for the line through $(1, 2, -1)$ and the origin. Sketch.

2. Find vector equations for the line through $(1, 2, -1)$ which is parallel to the y-axis. Sketch. What about parametric or symmetric equations?

3. Describe the line in the previous problem as the intersection of 2 planes. Is there an easy way to find another pair of planes whose intersection is this line?

4. Find vector and linear equations for the plane containing $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Sketch.

5. Find a normal vector for the plane in the last problem. Sketch.

Worksheet 5: Quadratic surfaces

Sketch the following quadratic surfaces in $\mathbb{R}^3$. Say what type of surface each is.

1. $x^2 + y^2 = z$
2. $x^2 + y^2 = z^2$
3. $x^2 + y^2 + 4z^2 = 1$
4. $x^2 - y^2 = 1$
5. $x^2 - y^2 = z$
6. $z = \sqrt{1 - x^2 - y^2}$
7. $x + y = z^2$

Worksheet 6: Space curves

1. Sketch the curve C given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$. Can you find another parametrization for C?

2. Let $\mathbf{r}(t) = \frac{\cos t}{t}\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + \frac{1}{t}\mathbf{k}$ for $0 < t < \infty$. Find $\lim_{t\to\infty} \mathbf{r}(t)$ if it exists. Sketch the curve.

3. Find two curves from $(0, 0, 0)$ to $(2, -1, -1)$. For each curve give vector and parametric equations, and sketch the curve.

4. Find two curves lying on the sphere $S : x^2 + y^2 + z^2 = 1$. For each curve give vector and parametric equations, and sketch the curve.

5. Let C be the intersection of $x^2 + y^2 = 1$ and $y + z = 1$. Find a parametric description for C.

Worksheet 7: Derivatives and Integrals of Vector Functions

For $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ a vector function, define the derivative $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$ and definite integral $\int_a^b \mathbf{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\rangle$ componentwise (when these quantities exist).

1. (a) Give a physical interpretation for $\mathbf{r}'(t)$ in terms of motion.

   (b) Give a physical interpretation for $\int_a^b \mathbf{r}(t)$ in terms of motion.

2. Let $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. Compute $\mathbf{r}'(0)$. Sketch the graph of $\mathbf{r}(t)$ and graph the vector $\mathbf{r}'(0)$ starting at the base point $\mathbf{r}(0)$.

3. Explain why $\mathbf{r}'(a)$ is a tangent vector to the curve given by $\mathbf{r}(t)$ at $t = a$ (for general $\mathbf{r}(t)$ differentiable at $t = a$).

4. Compute the unit tangent vector at $t = 0$ for $\mathbf{r}(t)$ as in Problem 1. Why might it be useful to look at a unit tangent vector rather than just a tangent vector?

5. (a) Give an example of a vector function $\mathbf{r}(t)$ which is not continuous at $t = a$.

   (b) Give an example of a vector function $\mathbf{r}(t)$ which is continuous at $t = a$ but not differentiable at $t = a$.

   (c) For each of the two above examples explain why there is or isn't a tangent vector.

Worksheet 8: Arc Length, Curvature and Motion

1. Compute the arc length of $C : \mathbf{r}(t) = \frac{1}{2}e^t\mathbf{i} + \cos e^t\mathbf{j} + \sin e^t\mathbf{k}$, $0 \leq t \leq 1$.

2. (i) Sketch the ellipse given by $E : \mathbf{r}(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$.

   (ii) The curvature $\kappa(t) = |\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}|$ where $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. Determine where the curvature of E is maximal and minimal.

3. (i) Check that a circle has constant curvature.

   (ii) Determine the curvature of a circle of radius c.

4. Suppose a standing rocket launches at time $t = 0$ from the origin and maintains constant acceleration in the direction of $\mathbf{a} = \langle 1, 2, 3\rangle$. Find the position and velocity of the rocket at time $t = 2$. What is the net distance traveled?

5. Say an object of mass m moves in a circle with position vector $\mathbf{r}(t) = a\cos kt\mathbf{i} + a\sin kt\mathbf{j}$.

   (i) Compute the force vector $\mathbf{F}(t) = m\mathbf{a}(t)$.

   (ii) Check that the force vector is always pointing toward the center of the circle (centripetal force).

Worksheet 9: Functions of several variables

1. For $f(x, y) = 6 - 3x - 2y$
   (a) find the domain (include sketch) and range;
   (b) sketch some level curves;
   (c) sketch the graph.
2. For $f(x, y) = \sqrt{9 - x^2 - y^2}$
   (a) find the domain (include sketch) and range;
   (b) sketch some level curves;
   (c) sketch the graph.
3. For $f(x, y) = \sqrt{x^2 + y^2 - 1}$
   (a) find the domain (include sketch) and range;
   (b) sketch some level curves;
   (c) sketch the graph.
4. Find and sketch the domain of $f(x, y) = \frac{\sqrt{x^2+y^3}}{x^2+3x-7}$.
5. Find and sketch the domain of $f(x, y) = \exp\left(\frac{x+y}{xy}\right)$.
6. Determine, as well as you are able, the level surfaces for $f(x, y, z) = x^2 + y^2 - z^2$.

Worksheet 10a: Limits and Continuity I

1. What does it mean for a function of a single variable $f(x)$ to have a limit as $x \to a$? What about to be continuous?

2. How would you try to define the notion of limits and continuity for a function $f(x, y)$ of two variables? Three variables?

3. Can you give an example of a function $f(x, y)$ whose limit does not exist at the origin?

4. Can you give an example of another function $f(x, y)$ which is not continuous at the origin?

5. Should $(x, y) \to 0$ mean the same thing as $x \to 0$ and $y \to 0$?

6. Should $\lim_{(x,y)\to 0} f(x, y) = \lim_{x\to 0} f(x, 0) = \lim_{y\to 0} f(0, y)$ if the limits exist?

Worksheet 10b: Limits and Continuity II

1. Let $f(x,y) = \frac{\sin x + \cos y}{x+y+1}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or determine it does not exist.

2. Let $f(x,y) = \frac{x^2 y}{x^2+y^2}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or determine it does not exist.

3. Let $f(x,y) = \frac{xy}{x^2+y^2}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or determine it does not exist.

4. Does there exist a function $f(x,y)$ such that $f(x,y) \to 0$ as $(x,y) \to 0$ along any line through the origin but $\lim_{(x,y)\to 0} f(x,y) \neq 0$.

Worksheet 11a: Partial Derivatives I

1. Recall what the definition of the derivative is for a function $f(x)$ of one variable. What does it mean geometrically? Kinematically (in terms of motion)?

2. Can you think of a geometric analogue of derivative for a function $f(x, y)$ of two variables?

3. Can you think of a kinematic analogue of derivative for a function $f(x, y)$ of two variables?

4. How might you try to define derivatives for functions of two (or more) variables?

Worksheet 11b: Partial Derivatives II

1. Let $f(x, y) = 3x + x^2y^3 - 2y^2$. Find the first and second partial derivatives of f. Check $f_{xy} = f_{yx}$.

2. Let $f(x, y) = x\sin(xy)$. Find the first and second partial derivatives of f. Check $f_{xy} = f_{yx}$.

3. Find the first partial derivatives of $f(x, y, z) = e^{xy}\ln z$.

4. For $f(x, y) = 3x + x^2y^3 - 2y^2$, determine the tangent lines of the cross sections of $z = f(x, y)$ for $x = 0$ and $y = 0$ at the origin. Determine the tangent plane to $z = f(x, y)$ at the origin.

Worksheet 12: Tangent planes and directional derivatives

1. Consider the sphere $S : x^2 + y^2 + z^2 = 1$.

   (i) At what points does S have a horizontal tangent plane?

   (ii) At what points does S have a vertical tangent plane?

   (iii) Find an equation for the tangent plane to S at $(1/\sqrt{2}, 0, 1/\sqrt{2})$.

2. (i) Find an equation for the tangent plane to $z = f(x, y) = x^2y^3$ at $(1, 1, 1)$.

   (ii) Use this tangent plane to approximate $f(1.1, 1.1)$.

3. Let $f(x, y) = x^2 + 4y^2$. Find the directional derivatives $D_{\mathbf{u}}f(x, y)$ at $(x, y) = (1, 1)$ in the direction with angle θ from the positive x-axis (counterclockwise) where (i) $\theta = 0$; (ii) $\theta = \frac{\pi}{2}$; and (iii) $\theta = \frac{\pi}{4}$.

4. With $f(x, y)$ as the in previous problem, compute the gradient $\nabla f(x, y)$. Use this to recompute the directional derivatives from the previous problem.

5. Let $f(x, y, z) = xy^2 + yz^3 + zx^5$.

   (i) In what direction does $f(x, y, z)$ increase the fastest at $(1, 1, 1)$?

   (ii) What is this maximal rate of increase?

6. Show that every line normal to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the origin.

Worksheet 13: Extreme Values

1. Find the critical points of $f(x, y) = 3x^4 + 3y^4 - 12xy + 1$ and classify them as having local minima or local maxima or being saddle points. Determine the absolute minimum and maximum of $f(x, y)$ if they exist.

2. Find the distance from the point $(1, 2, 3)$ to the plane $x + y + z = 10$.

3. Find the maximum volume of a rectangular box such that the width, depth and height of the box sum to 12m.

4. Let R be the region in $\mathbb{R}^2$ consisting of (x, y) such that $0 \leq x, y \leq 1$. Can you exhibit a function $f : R \to \mathbb{R}^2$ such that f has no absolute maximum? Can you guess sufficient conditions on f to guarantee that is has or doesn't have an absolute maximum?

5. Find the absolute maximum and minimum of $f(x, y) = xy$ on the region $x^2 + y^2 \leq 1$. What about the region $x^2 + y^2 < 1$.

6. Find the absolute maximum and minimum of $f(x, y) = x^2 - 2xy + y$ on the region of $\mathbb{R}^2$ given by $0 \leq x \leq 3$ and $0 \leq y \leq 2$.

Worksheet 14: Double Integrals I

1. Compute $\int\int_R (x - 3y^2)\, dA$ where $R = [0, 3] \times [1, 2]$.

2. Find the volume of the solid bounded by $x^2 + 2y^2 + z = 16$, $x = 2$, $y = 2$, and the 3 coordinate planes.

Worksheet 15: Review

Warning: This is not a comprehensive review for Exam 1. See the suggested problems on the exams page for a much more comprehensive set of practice problems.

1. True or False: For any a, b, c, the planes $ax + by + cz = 0$ and $ax + by + cz = 1$ are parallel.
2. True or False: For any a, b, the distance between the planes $x + y + z = a$ and $x + y + z = b$ is $|a - b|$.
3. True or False: $\mathbf{i} \times \mathbf{j} = \mathbf{j} \times \mathbf{i}$.
4. True or False: The cross product of two nonzero vectors can be 0.
5. Explain what the dot product tells you geometrically.
6. Give an example of a surface in $\mathbb{R}^3$ which has a critical point at the origin but neither a local max nor min there.
7. Find the domain and range of $f(x, y) = \sqrt{x^2 - y^2}$. Sketch the graph.
8. What is the direction of fastest decrease of $f(x, y) = x^2y$ at (1,2)? What is the rate of decrease in this direction?
9. Find the equation for the tangent plane to $z = x^2y$ at $(1, 2, 2)$.
10. Find an equation for the tangent line to the curve $\mathbf{r}(t) = \langle t^2 - t, \sqrt{t}, t + 1\rangle$ at $t = 1$.
11. Suppose a particle initially at rest at the origin in $\mathbb{R}^3$ moves with acceleration $\mathbf{a}(t) = \langle t, t^2, t^3\rangle$. Where is the particle at time $t = 1$?
12. Determine the angle between the planes $y + z = 1$ and $y = 2z$.

Worksheet 16: Double Integrals II

1. Give examples of regions that are (i) type I but not type II; (ii) type II but not type I; (iii) type I and type II; (iv) neither type I nor type II.

2. Find the volume of the tetrahedron bounded by $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

3. Compute $\int\int_R xy^2 \, dA$, where R is region bounded by $y = x - 1$ and $y^2 = 2x + 6$.

4. Compute $\int\int_R \sin(y^2) \, dA$, where R is region enclosed by the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$.

Worksheet 17: Double Integrals in Polar Coordinates

1. Find the volume of the region bounded by $z = x^2 + y^2$ and $z^2 = x^2 + y^2$.
2. Use a double integral to compute the area in $\mathbb{R}^2$ of one loop of the curve $r = \cos\theta$.
3. Use a double integral to compute the area in $\mathbb{R}^2$ enclosed by $r = \frac{\theta}{\pi}$, $0 \le \theta \le 3\pi$ and the line segment from $(-1, 0)$ to $(-3, 0)$.
4. (i) Compute volume under $e^{-x^2-y^2}$ over all of $\mathbb{R}^2$ by using polar coordinates.

   (ii) Use (i) to compute $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Worksheet 18: Surface Area and Triple Integrals

1. Compute $\int\int\int_E xyz^2\, dV$ where $E = [0,1] \times [-1,2] \times [0,3]$.

2. Using a triple integral, compute the volume of the solid tetrahedron bounded by the 3 coordinate planes and $x + y + z = 1$.

3. Find the surface area of the portion of $z = x^2 + 2y$ that lies above the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

4. Find the surface area of the portion of $z = x^2 + y^2$ lying under $z = 9$.

5. Let E be the region in $\mathbb{R}^3$ bounded by $y = x^2 + z^2$ and $y = 4$. Compute $\int\int\int_E \sqrt{x^2 + z^2}\, dV$.

Worksheet 19: Cylindrical coordinates

1. True or False: If we restrict $0 \leq \theta < 2\pi$, there is a unique cylindrical coordinate (r, θ, z) representing a given (x, y, z). Explain.

2. Graph the objects described by each of the following cylindrical equations:

   (i) $z = r$;

   (ii) $z = r^2$;

   (iii) $z^2 = r^2$;

   (iv) $z = \theta$;

   (v) $r = \theta$.

3. Let E be the solid cone bounded between $z = \sqrt{x^2 + y^2}$ and $z = 2$. Compute $\int\int\int_E (x^2 + y^2) dV$.

4. Let E be top-half of the solid sphere given by $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$. Compute $\int\int\int_E z dV$.

Worksheet 20: Spherical coordinates

1. (i) Under what conditions will the spherical coordinates (ρ, θ, ϕ) and (ρ', θ', ϕ') represent the same point? (You may assume $\rho, \rho' \geq 0$.)

   (ii) What restrictions can you place on ρ, θ, ϕ so that (x, y, z) determines a unique spherical coordinate (ρ, θ, ϕ)?

2. Describe the following objects using spherical coordinates:

   (i) the sphere of radius 3 centered at $(0, 0, 0)$;

   (ii) the portion of the sphere $x^2 + y^2 + z^2 = 2$ lying in the octant $x \geq 0$, $y \geq 0$, $z \leq 0$;

   (iii) the plane $x = y$;

   (iv) the (half) cone $z = \sqrt{x^2 + y^2}$;

   (v) the x-axis;

   (vi) the point $(x, y, z) = (-1, 1, -1)$.

3. Try to sketch the following objects given in spherical coordinates:

   (i) $0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \theta \leq \frac{\pi}{2}$ and $1 \leq r \leq 2$;

   (ii) $\rho = \phi$, $0 \leq \phi \leq \frac{\pi}{2}$;

   (iii) $\rho = \theta$, $0 \leq \theta \leq \frac{\pi}{2}$;

   (iv) $\theta = \phi$ and $\rho = 1$

4. Let B be the ball $x^2 + y^2 + z^2 \leq 1$. Compute $\int \int \int_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$.

5. Use a triple integral in spherical coordinates to compute the volume of the solid S bounded between the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = z$.

Worksheet 21: Vector Fields and Line Integrals I

1. Sketch the following vector fields and some flow lines:

   (i) $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$;

   (ii) $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$;

   (iii) $\mathbf{F}(x, y) = y\mathbf{i} + \sin x\mathbf{j}$;

   (iv) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$.

   (v) $\mathbf{F}(x, y, z) = \cos z\mathbf{i} + \sin z\mathbf{j} + z\mathbf{k}$.

2. For the following functions, sketch a few level curves as well as the gradient fields along these curves.

   (i) $f(x, y) = xy$;

   (ii) $f(x, y) = x^2 + y^2$.

3. Let C be the line segment in $\mathbb{R}^2$ from $(0, 0)$ to $(2, 1)$. Verify that the line integral $\int_C (x + 2y)\, ds$ gives the area of the triangle in $\mathbb{R}^3$ with vertices $(0, 0, 0)$, $(2, 1, 0)$ and $(2, 1, 4)$.

4. Let $h, R > 0$ and C be the circle $x^2 + y^2 = R^2$ in $\mathbb{R}^2$. Verify that the line integral $\int_C h\, ds$ gives the surface area of the cylinder $x^2 + y^2 = R^2, 0 \leq z \leq h$ in $\mathbb{R}^3$.

5. Let C be the semicircle $y = \sqrt{1 - x^2}$ in $\mathbb{R}^2$. Compute $\int_C (2 + x^2 y)\, ds$.

6. Redo one of the previous integrals using a different parametrization for C, and check the result is the same.

Worksheet 22: Line Integrals II

1. Let C be the portion of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$ (moving counter-clockwise). Let $f(x, y) = x$. Compute the following line integrals:

   (i) $\int_C f(x, y)\, ds$;

   (ii) $\int_{-C} f(x, y)\, ds$;

   (iii) $\int_C f(x, y)\, dx$;

   (iv) $\int_{-C} f(x, y)\, dx$;

   (v) $\int_C f(x, y)\, dy$.

2. What relations are there between the integrals in the previous problem? Can you explain why? Are there any relations between the line integrals with respect to arclength and those with respect to x and y?

3. Consider the vector field $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$.

   (i) Compute the work done by $\mathbf{F}$ along the curve C from Problem 1.

   (ii) Compute the work done by $\mathbf{F}$ along the curve C', where C' is the portion of the parabola $x = y^2$ from $(0, 0)$ to $(1, 1)$.

   (iii) Sketch the vector field and the curves C and C'. Explain what the answers to (i) and (ii) mean and why they make sense graphically.

4. Let $\mathbf{F}$ be as in the previous problem.

   (i) Explain why the work done by $\mathbf{F}$ along any circle $x^2 + y^2 = r^2$ must be 0.

   (ii) Explain why the work done by $\mathbf{F}$ along any square centered at the origin must also be 0.

   (iii) Can you find a closed path C (a curve whose start and end points are the same) such that the work done by $\mathbf{F}$ along C is nonzero?

Worksheet 23: The Fundamental Theorem for Line Integrals

1. Let C be the path in $\mathbb{R}^2$ beginning at $(0,0)$, travelling to $(1,1)$ along $y = \sqrt{x}$, then travelling to $(2,4)$ along $x = \sqrt{y}$, then travelling to and ending $(4,0)$ in a straight line. Find the work done by $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$ along C.

2. Sketch each of the following vector fields, and using the sketch. explain why each is conservative or not. Realize those that are conservative as gradient vector fields (find f so that $\mathbf{F} = \nabla f$).

   (a) $\mathbf{F}(x,y) = \mathbf{i} + 2\mathbf{j}$

   (b) $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j}$

   (c) $\mathbf{F}(x,y) = y\mathbf{i} - x\mathbf{j}$

   (d) $\mathbf{F}(x,y,z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

3. True or False: If $\mathbf{F}$ is conservative, then the work done along C is the same as the work done along $-C$ for any path C.

4. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. Determine a criterion to determine whether $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-C} \mathbf{F} \cdot d\mathbf{r}$ for all closed curves C in $\mathbb{R}^2$.

Worksheet 24: Green's Theorem

1. Using the "$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$" criterion, determine whether the following fields are gradient fields or not. For those that are, realize them as gradients.

   (a) $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$

   (b) $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$

2. Let C be the path in $\mathbb{R}^2$ obtained by traversing the rectangle with vertices (in this order): $(0, 0)$, $(3, 0)$, $(3, 2)$, $(0, 2)$. Using Green's theorem, compute $\int_C (x - y)\,dx + (x + y)\,dy$.

3. Let C_1 be the path in $\mathbb{R}^2$ obtained by traversing the 3 line segments from $(0, 0)$ to $(3, 0)$, from $(3, 0)$ to $(3, 2)$, and from $(3, 2)$ to $(0, 2)$. Using your answer to the previous problem, compute $\int_{C_1} (x - y)\,dx + (x + y)\,dy$ by evaluating only a single other line integral. (So don't compute this directly as a sum of 3 line integrals.)

Worksheet 25: Curl and Div

1. For each of the following vector fields $\mathbf{F}$, sketch $\mathbf{F}$, compute curl $\mathbf{F}$ and div $\mathbf{F}$. Explain what features of $\mathbf{F}$ are captured by curl $\mathbf{F}$ and div $\mathbf{F}$.

   (a) $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

   (b) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

2. Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on $\mathbb{R}^2$, which we view also as a vector field on $\mathbb{R}^3$ via $\mathbf{F}(x, y, z) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + 0\mathbf{k}$.

   (a) compute curl $\mathbf{F}$

   (b) use (a) to rewrite the statement Green's theorem in terms of curl $\mathbf{F}$

3. (a) For the curls you computed in the previous problems, check that div(curl $\mathbf{F}$) $= 0$

   (b) Explain why for any vector field $\mathbf{F}$ on $\mathbb{R}^3$, one should have div(curl $\mathbf{F}$) $= 0$.

Worksheet 26: Parametric Surfaces

1. Parametrize the following surfaces in $\mathbb{R}^3$:

   (a) The quadrilateral with vertices $(0,0,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,2)$.

   (b) The triangle with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,1)$.

   (c) The portion of the paraboloid $z = x^2 + y^2$ lying under the plane $z = 4$.

   (d) The portion of the cylinder $x^2 + y^2 = 1$ lying between $z = 0$ and $z = 4$.

2. Suppose $\mathbf{r} : D \to S$ is a parametrization of the surface S in $\mathbb{R}^3$, where D is a region in $\mathbb{R}^2$. What reasonable condition do you need on the parametrization for the area of S to be given by $\int\int_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$? (Think about when a parametrization for a curve is not suitable for computing arclength.)

3. Using your parametrization for 1(d), compute the surface area of the cylinder in 1(d) with the formula $\int\int_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$.

Worksheet 27: Surface Integrals

1. Let S be the sphere $x^2+y^2+z^2=1$ with positive orientation. For each of the following vector fields, determine whether the flux over S should be positive, negative or zero. (Don't compute any integrals.)

   (a) $\mathbf{F}(x,y,z) = \mathbf{i}+\mathbf{j}+\mathbf{k}$

   (b) $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

   (c) $\mathbf{F}(x,y,z) = -x\mathbf{i}-y\mathbf{j}$

   (d) $\mathbf{F}(x,y,z) = y\mathbf{i}-x\mathbf{j}$

2. True or False: Let $\mathbf{F}$ be a conservative vector field on $\mathbb{R}^3$. The flux of $\mathbf{F}$ over any S closed oriented surface is 0.

3. Which of the following things can you compute with with Stokes' theorem? Explain.

   (a) the flux of a vector field over a closed oriented surface

   (b) the flux of a vector field over a non-closed surface

   (c) the work done by a vector field over a simple closed curve

   (d) the work done by a vector field over a simple non-closed curve

   (e) the area of a closed oriented surface

   (f) the area of a non-closed oriented surface

4. Let S be the portion of $z=x+y^2$ lying above $D=[0,1]\times[0,2]$. Compute $\int\int_S y\,dS$.

5. Let S be the surface from the previous problem, and C be its boundary. Using Stokes' Theorem, compute the the work done by $\mathbf{F}(x,y,z)=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$ along C.

Worksheet 28: Stokes' and Divergence Theorems

1. Which of the following things can you compute Stokes' Theorem? Which can you compute with the Divergence Theorem? Explain.

   (a) the flux of a vector field over a closed oriented surface

   (b) the flux of a vector field over a non-closed surface

   (c) the work done by a vector field over a simple closed curve

   (d) the work done by a vector field over a simple non-closed curve

   (e) the area of a closed oriented surface

   (f) the area of a non-closed oriented surface

   (g) the volume of a solid

2. Let S be the portion of $z = x + y^2$ above $D = [0,1] \times [0,2]$, and C be its boundary. Using Stokes' Theorem, compute the the work done by $\mathbf{F}(x,y,z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ along C.

3. Let S be the surface of the tetrahedron with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(1,1,1)$. Compute the flux of $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ over S.

Worksheet The-One-After-The-Last[1]: Review (Ch 12 – Ch 15)

1. True or False: The planes $x+2y+3z=0$ and $2x+4y+6z=8$ are parallel. Explain.
2. Write an equation for the plane through $(1,1,1)$, $(1,2,3)$ and $(0,1,-1)$.
3. Sketch a graph of $x^2+z^2=e^{-y}$.
4. Write an equation for the tangent plane to $z=x+y^2$ at $(-1,2,3)$.
5. Find all local mins, maxes and saddle points for $z=f(x,y)=x^3-3xy+y^3$.
6. Describe the line $\mathbf{r}(t)=\langle t,t,t\rangle$ with equations using (i) Cartesian coordinates; (ii) cylindrical coordinates; and (iii) spherical coordinates.
7. Compute the volume of the region contained in the first octant of $\mathbb{R}^3$ which lies inside $x^2+y^2=1$, above $z=0$ and below $z=2x+y+3$.
8. Let E be the solid part of the half-cone $z=\sqrt{x^2+y^2}$ contained in the sphere $x^2+y^2+z^2=4$. Compute $\int\int\int_E \sqrt{x^2+y^2+z^2}\,dV$. (Remark: this represents the average distance of a point of E from the vertex times the volume of E.)

[1]Post-ultimate?