Applied Algebra (MATH 4383/5383) Spring 2009
Exam 2

Due: Friday May 1, 2009 (in class)

Instructions: You may use your text and course notes but no other references. You may not collaborate on the exam, however, you are free to ask me questions about these problems.

1. Let $\mathcal{D} = (\mathcal{B}, \mathcal{P})$ be a projective plane of order n. Show that removing any block and all the points which occur in that block yields an affine plane of order n.

2. Let $\mathcal{D} = (\mathcal{B}, \mathcal{P})$ be the projective plane of order 3. Let $\mathcal{I}$ be the incidence matrix for $\mathcal{D}$ and let $\mathcal{C} = \mathcal{C}_3(\mathcal{D})$ be the associated linear ternary code, i.e., the code over $\mathbb{F}_3$ whose generator matrix is $\mathcal{I}$. Write down $\mathcal{I}$ and determine the $[n, k, d]_3$ parameters for $\mathcal{C}$.

3. Draw a 4×4-grid which represents $\mathcal{P} := \mathbb{Z}/4 \times \mathbb{Z}/4$. Consider the set of lines $\mathcal{B}$ given by the solutions to the linear equations $\{(x, y) \in \mathcal{P} | ax + by \equiv c \mod 4\}$ where $a, b, c \in \mathbb{Z}/4$. Which of the following properties fail:

(i) Any two points lie in exactly one line;
(ii) Any two lines interesect in at most one point.

Draw appropriate lines on the grid to justify your answer.

4. Draw a 4×4-grid which represents $\mathcal{P} = \mathbb{F}_4 \times \mathbb{F}_4$ (label the rows and columns), and define the lines in the usual way. This *is* an affine plane of order 4. (Compare with the previous problem.) Draw all the lines which pass through the origin.

5. For each $1 \leq n \leq 25$, determine if $n = x^2 + y^2$ for some $x, y \in \mathbb{Z}$. For which of these n can you say there is or is not a projective plane of order n using (i) the Bruck-Ryser–Chowla theorem and (ii) the construction of a projective plane over a field?

6. (Classification of cubics) Consider $\mathbb{P}^2(\mathbb{R})$. We saw in lecture that each (affine) line has one point at infinity, and a quadratic could have either 0 (circle), 1 (parabola) or 2 (hyperbola) points at infinity. Topologically, however, they all look like circles.

Fact: Any smooth irreducible cubic can be transformed via a change of variables in to one of the following forms

$$y^2 = x(x-1)(x-w), \quad w > 1 \qquad (1)$$

$$y^2 = x(x^2 + kx + 1), \quad -2 < k < 2. \qquad (2)$$

Let C_1 be a cubic of the first type with $w = 2$ and C_2 be a cubic of the second type with $k = 0$. Sketch a graph of C_1 and C_2 in the affine plane. Homogenize these equations and determine the points at infinity on each of these curves. What do the projective curves C_1 and C_2 look like topologically?

7. Let $\mathbb{P}^1(\mathbb{C})$ be the complex projective line, using the definition of projective space in class. I.e., in homogenous coordinates,

$$\mathbb{P}^1(\mathbb{C}) = \{(z : w) | z, w \in \mathbb{C} \text{ not both } 0\}.$$

Here as usual we are identifying $(z : w) = (\lambda z : \lambda w)$ for $\lambda \neq 0$. Define a map $\iota : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ which embeds the usual complex plane $\mathbb{C} \simeq \mathbb{R}^2$ (or affine complex line depending on your terminology) into $\mathbb{P}^1(\mathbb{C})$ in terms of homogenous coordinates. The remaining points $\mathbb{P}^1(\mathbb{C}) - \iota(\mathbb{C})$ are the points at infinity. What are they in terms of homogenous coordinates?

What do you think $\mathbb{P}^1(\mathbb{C})$ looks like topologically? Is $\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}^2(\mathbb{R})$?

Bonus. Exercise 7.8, p. 100 from the textbook.