MATH 2433
Summer 2017
Exam 1
6/19/2017
Time Limit: 60 Minutes

Name: ____________________

This exam contains 7 pages (including this cover page) and 6 questions.
Total of points is 12.

Please write down all the explanations as well as your final answer. No calculators are allowed.

Grade Table (for teacher use only)

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
Total:	12	

1. (2 points) Determine whether each of the following statements is true or false. (You may write "T" next to a true statement and "F" next to a false statement.)

F 1) $\lim_{n\to\infty} a_n = L$ if for any $\epsilon > 0$, the entries a_n's are in the interval $(L-\epsilon, L+\epsilon)$ for all n's. for k, k+1, k+2, ... starting at some k

F 2) $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is an alternating series. it is required that $a_n > 0$

F 3) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges. counter e.g. $\Sigma \frac{1}{n}$ diverges

F 4) If a series converges, then it is absolutely convergent. counter e.g. $\Sigma (-1)^{n+1} \frac{1}{n}$ is convergent yet $\Sigma \frac{1}{n}$ is divergent.

F 5) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $a_n < b_n$ for all n. if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges. a_n needs to be positive for all n's

F 6) $\sum_{n=1}^{\infty}\sin(3n)$ is an alternating series.
e.g. $n = 31$. $29\pi \approx 91.06 < 93 < 94.2 \approx 30\pi$ therefore $\sin(3\cdot 31)$ is negative but $(-1)^{31+1} = 1$ is positive

F 7) Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$ for all n's. If $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = 1$, then $\sum_{n=1}^{\infty} a_n = \infty$.
The root test is inconclusive when $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$

3. (2 points) Show that the following series is convergent

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{3^n + 4^n}$$

You may use any test you prefer.

$$\frac{|\sin n|}{3^n+4^n} \le \frac{1}{3^n+4^n}$$

let $a_n = \frac{|\sin n|}{3^n+4^n}$, $b_n = \frac{1}{3^n+4^n}$, $c_n = \frac{1}{4^n}$

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{3^n+4^n}{4^n} = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n + 1 = 1$$

since Σc_n is ~~conv~~ a geometric series with common ratio $r = \frac{1}{4} < 1$

Σc_n is convergent

hence Σb_n is convergent by limit comparison test (with Σc_n)

Σa_n is convergent by comparison test (with Σb_n)

Method 2: $\frac{|\sin n|}{3^n+4^n} \le \frac{1}{3^n+4^n} \le \frac{1}{4^n}$

$\Sigma \frac{1}{4^n}$ is convergent as a geometric series with common ratio $r = \frac{1}{4} < 1$

$\Sigma \frac{1}{4^n}$ is convergent

therefore $\Sigma \frac{|\sin n|}{3^n+4^n}$ is convergent by Comparison Test

2. (2 points) Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

$$= (A+B)n + 2A$$

$$\begin{cases} A+B=0 \\ 2A=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$$

$$\frac{1}{n(n+2)} = \frac{1}{2}\cdot\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$S_n = a_1 + \cdots + a_n = \frac{1}{2}\left(1-\cancel{\frac{1}{3}}\right) + \frac{1}{2}\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)$$

$$+ \cdots + \frac{1}{2}\left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \frac{1}{2}\left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}$$

4. (2 points) Show that the following series is divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

You may use any test you prefer.

Use limit comparison Test

$$a_n = \frac{1}{n^{1+\frac{1}{n}}}$$

$$b_n = \frac{1}{n}$$

$$\frac{b_n}{a_n} = \frac{1}{n} \cdot n^{1+\frac{1}{n}} = n^{\frac{1}{n}}$$

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} (e^{\ln n})^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} e^{\frac{\ln n}{n}}$$

$$= e^{\lim_{n\to\infty} \frac{\ln n}{n}}$$

$$\overset{L'}{=} e^{\lim_{n\to\infty} \frac{\frac{1}{n}}{1}} = e^0 = 1$$

therefore $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together

by p-series test. $\sum b_n$ diverges $(p=1)$

therefore $\sum a_n$ diverges

5. (2 points) Use the ratio test, show that the following series is convergent

$$\sum_{n=1}^{\infty} \frac{2^n(n+1)}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+2)}{(n+1)!} \cdot \frac{n!}{2^n \cdot (n+1)}$$

$$= \frac{n!}{(n+1)!} \cdot \frac{2}{1} \cdot \frac{n+2}{n+1}$$

$$= \frac{2}{n+1} \cdot \frac{n+2}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{2}{n+1} \cdot \lim \frac{n+2}{n+1}$$

$$= 0 \cdot \lim \frac{1+2/n}{1+1/n} = 0 < 1$$

therefore Σa_n is convergent

6. (2 points) Prove the following (The Integral Test:)
Let $\sum_{n=1}^{\infty} a_n$ be a series given by

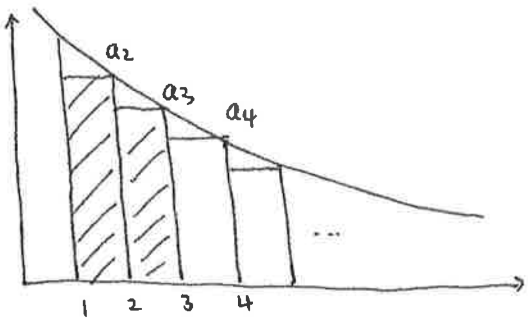
$$a_n = f(n)$$

for some function $f(x)$ such that $f(x) > 0$ for all $x > 0$ and $f(x)$ is decreasing.
Show that if the improper integral is convergent

$$\int_1^{\infty} f(x)\mathrm{d}x$$

then $\sum_{n=1}^{\infty} a_n$ is convergent.
Please include any picture(s) that may be helpful.

Proof.



$$S_n = a_1 + (a_2 + \cdots + a_n)$$

$$= a_1 + (f(2)\cdot 1 + \cdots + f(n)\cdot 1)$$

$= a_1 +$ total area of the first n rectangles

$$\leq a_1 + \int_1^{\infty} f(x)dx$$

Since $\int_1^{\infty} f(x)dx$ is convergent

$$\lim_{n\to\infty} S_n \leq a_1 + \int_1^{\infty} f(x)dx$$

therefore ~~the~~ $\lim_{n\to\infty} S_n$ is finite

Since $a_n > 0$, $\lim_{n\to\infty} S_n$ is convergent

$\sum_{n=1}^{\infty} a_n$ is convergent

MATH 2433
Summer 2017
Exam 2
7/3/2017
Time Limit: 60 Minutes

Name: ____________________

This exam contains 8 pages (including this cover page) and 6 questions. Total of points is 14.

Please write down all the explanations as well as your final answer. No calculators are allowed.

Grade Table (for teacher use only)

Question	Points	Score
1	4	
2	2	
3	2	
4	2	
5	2	
6	2	
Total:	14	

1. Consider the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x-2)^{2n}}{3^n \cdot \ln n}$$

(a) (2 points) Find the radius of convergence of the above series.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}(x-2)^{2(n+1)}}{3^{n+1}\ln(n+1)} \cdot \frac{3^n \ln n}{(-1)^n (x-2)^{2n}}$$

$$= \frac{(-1)^{n+1}(x-2)^{2n+2}}{3^{n+1}\ln(n+1)} \cdot \frac{3^n \ln n}{(-1)^n (x-2)^{2n}}$$

$$= \frac{(-1)}{3} \cdot \frac{(x-2)^2}{1} \cdot \frac{\ln n}{\ln(n+1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{-1}{3} \cdot \frac{(x-2)^2}{1} \cdot \frac{\ln n}{\ln n+1}\right|$$

$$= \lim_{n\to\infty} \frac{(x-2)^2}{3} \lim_{n\to\infty} \frac{\ln n}{\ln n+1}$$

$$\overset{L'}{=} \frac{(x-2)^2}{3} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{(x-2)^2}{3}$$

$$\frac{(x-2)^2}{3} < 1$$

$$(x-2)^2 < 3$$

$$|x-2| < \sqrt{3}$$

$$2-\sqrt{3} < x < 2+\sqrt{3}$$

(b) (2 points) Check if the series is convergent or divergent at the endpoints you found in part a).

At $x = 2+\sqrt{3}$

$$\sum_{n=2}^{\infty} \frac{(-1)^n (2+\sqrt{3}-2)^{2n}}{3^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{3}^{2n}}{3^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{3^n \ln n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$$

since $\frac{1}{\ln n} > 0$, $\frac{1}{\ln n}$ decreases, $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by Alternating Series Test

At $x = 2-\sqrt{3}$

$$\sum_{n=2}^{\infty} \frac{(-1)^n (2-\sqrt{3}-2)^{2n}}{3^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n (-\sqrt{3})^{2n}}{3^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{3^n \ln n}$$

is the same series as the above, therefore the series is convergent at $x = 2-\sqrt{3}$

2. (2 points) Find the MacLaurin series expansion of the following function

$$\int e^{-x^2} dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int e^{-x^2} dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!} + C$$

3. (2 points) Find the Maclaurin series expansion of the following function

$$\frac{x^3}{(1+x)^2}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \sum_{n=1}^{\infty} n(-1)^n x^{n-1}$$

$$-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} n(-1)^n x^{n-1}$$

$$\frac{1}{(1+x)^2} = -\sum_{n=1}^{\infty} n(-1)^n x^{n-1}$$

$$\frac{x^3}{(1+x)^2} = -x^3 \sum_{n=1}^{\infty} n(-1)^n x^{n-1}$$

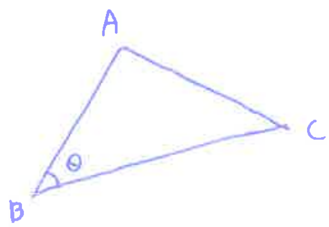
$$= -\sum_{n=1}^{\infty} n(-1)^n x^{n+2}$$

4. (2 points) Let

$$A = (3, 5, 2)$$
$$B = (0, -3, 8)$$
$$C = (2, 3, -4)$$

be three points in the space. Find the angle B inside the triangle ABC.

(You may leave your answer as an inverse cosine function.)



$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = (3, 8, -4)$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (2, 6, -12)$$

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| \cdot |\overrightarrow{BC}|} = \frac{3 \cdot 2 + 8 \cdot 6 + (-4) \cdot (-12)}{\sqrt{3^2 + 8^2 + (-4)^2} \cdot \sqrt{2^2 + 6^2 + (-12)^2}}$$

$$= \frac{6 + 48 + 48}{\sqrt{9 + 64 + 16} \cdot \sqrt{4 + 36 + 144}}$$

$$= \frac{102}{\sqrt{89}\sqrt{184}}$$

$$\theta = \cos^{-1} \frac{102}{\sqrt{89}\sqrt{184}} = \cos^{-1} \frac{51}{\sqrt{89}\sqrt{46}}$$

5. (2 points) Let **v** and **w** be two vectors satisfying the following

$$|\mathbf{v}| = 4$$
$$|\mathbf{w}| = 9$$
$$\mathbf{v} \cdot \mathbf{w} = -10$$

Find the angle between **v** and $2\mathbf{v} + 3\mathbf{w}$.

$$|2\vec{v}+3\vec{w}| = \sqrt{(2\vec{v}+3\vec{w})(2\vec{v}+3\vec{w})}$$

$$= \sqrt{4\vec{v}\cdot\vec{v} + 6\vec{w}\cdot\vec{v} + 6\vec{v}\cdot\vec{w} + 9\vec{w}\cdot\vec{w}}$$

$$= \sqrt{4|\vec{v}|^2 + 12\vec{v}\cdot\vec{w} + 9|\vec{w}|^2}$$

$$= \sqrt{4\cdot 4^2 + 12(-10) + 9\cdot 9^2}$$

$$= \sqrt{64 - 120 + 729} = \sqrt{673}$$

$$\cos\theta = \frac{\vec{v}\cdot(2\vec{v}+3\vec{w})}{|\vec{v}|\cdot|2\vec{v}+3\vec{w}|}$$

$$= \frac{2|\vec{v}|^2 + 3\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|2\vec{v}+3\vec{w}|}$$

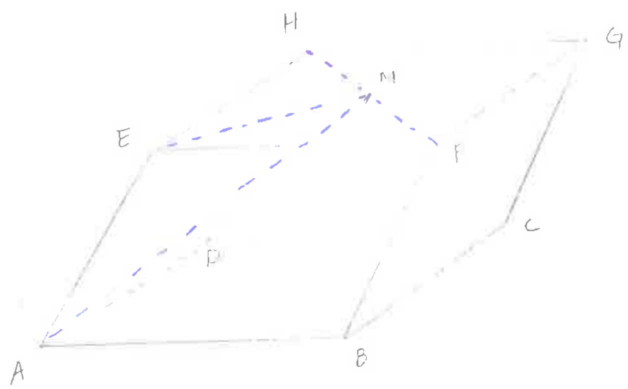
$$= \frac{2\cdot 4^2 + 3\cdot(-10)}{4\cdot\sqrt{673}}$$

$$= \frac{32-30}{4\cdot\sqrt{673}} = \frac{1}{2\cdot\sqrt{673}}$$

$$\theta = \cos^{-1}\frac{1}{2\sqrt{673}}$$

6. (2 points) In the picture below, M is the center of the parallelogram $EHGF$ on the top face. Express the vector $\overrightarrow{AM}$ as a linear combination of $\overrightarrow{AB}$, $\overrightarrow{AE}$ and $\overrightarrow{AD}$. That is, find real numbers α, β and γ such that

$$\overrightarrow{AM} = \alpha\overrightarrow{AB} + \beta\overrightarrow{AE} + \gamma\overrightarrow{AD}$$



Since M is the midpoint of HF

$$\overrightarrow{EM} = \frac{1}{2}(\overrightarrow{EH} + \overrightarrow{EF})$$

$$\left(\overrightarrow{EM} = \frac{1}{2}\overrightarrow{EG} = \frac{1}{2}(\overrightarrow{EH} + \overrightarrow{HG}) = \frac{1}{2}(\overrightarrow{EH} + \overrightarrow{EF}) \right)$$

$$\overrightarrow{AM} = \overrightarrow{AE} + \overrightarrow{EM}$$
$$= \overrightarrow{AE} + \frac{1}{2}(\overrightarrow{EH} + \overrightarrow{EF})$$
$$= \overrightarrow{AE} + \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{AB})$$

$\alpha = \frac{1}{2}$, $\beta = 1$, $\gamma = \frac{1}{2}$

MATH 2433
Summer 2017
Exam 3
7/19/2017
Time Limit: 60 Minutes

Name: ______________________

This exam contains 8 pages (including this cover page) and 7 questions.
Total of points is 14.

Please write down all the explanations as well as your final answer. No calculators are allowed.

Grade Table (for teacher use only)

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
Total:	14	

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1. (2 points) Let **v** and **w** be vectors such that

$$|\mathbf{v}| = 10$$
$$|\mathbf{w}| = 8$$
$$\theta = \frac{\pi}{3}$$

where θ is the angle between **v** and **w**.
Calculate

$$(\mathbf{v} + 3\mathbf{w}) \cdot (2\mathbf{v} - \mathbf{w}) \times (\mathbf{v} + \mathbf{w})$$

$(\vec{v} + 3\vec{w}) \cdot (2\vec{v} - \vec{w}) \times (\vec{v} + \vec{w})$

$= \cancel{(\vec{v} \cdot 2\vec{v} + 3\vec{w} \cdot 2\vec{v} - \vec{v} \cdot \vec{w} - 3\vec{w} \cdot \vec{w}) \times (\vec{v} + \vec{w})}$

$= \cancel{\vec{v} \cdot 2\vec{v} \times}$

$(\vec{v} + 3\vec{w}) \cdot (2\vec{v} \times \vec{v} - \vec{w} \times \vec{v} + 2\vec{v} \times \vec{w} - \vec{w} \times \vec{w})$

$= (\vec{v} + 3\vec{w}) \cdot (-\vec{w} \times \vec{v} + 2\vec{v} \times \vec{w})$ since $\vec{v} \times \vec{v} = \vec{w} \times \vec{w} = \vec{0}$

$= -\vec{v} \cdot \vec{w} \times \vec{v} - 3\vec{w} \cdot \vec{w} \times \vec{v}$
$+ 2\vec{v} \cdot \vec{v} \times \vec{w} + 6\vec{w} \cdot \vec{v} \times \vec{w}$

since $\vec{w} \times \vec{v}$ or $\vec{v} \times \vec{w}$
is perpendicular to both $\vec{v}$ and $\vec{w}$
each term is 0.

the answer = 0.

2. (2 points) Determine if the following four points lie on the same plane.

$$A(2,4,1)$$
$$B(0,4,-2)$$
$$C(1,1,3)$$
$$D(1,2,-1)$$

A, B, C, D lie on the same plane

if and only if the volume of the parallelopiped

formed by $\vec{AB}, \vec{AC}, \vec{AD}$ is 0.

$\vec{AB} = (-2, 0, -3)$

$\vec{AC} = (-1, -3, 2)$

$\vec{AD} = (-1, -2, -2)$

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & 2 \\ -1 & -2 & -2 \end{vmatrix} = (6-(-4))\vec{i} - (2-(-2))\vec{j} + (2-3)\vec{k} = (10, -4, -1)$$

$$\text{Vol} = |\vec{AB} \cdot \vec{AC} \times \vec{AD}| = |(-2, 0, -3) \cdot (10, -4, -1)|$$

$$= |-20+3| \neq 0$$

therefore A, B, C, D do not lie on the same plane

3. (2 points) Let l be the line that passes through the points $A(2,1,4)$ and $B(0,-7,3)$. Let p be the plane that contains the point $C(3,5,1)$ and is perpendicular to Line l, find the equation of the plane p.

$\overrightarrow{AB} = (-2, -9, -1)$ is a normal vector of the plane

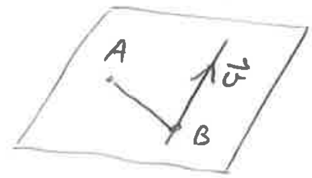
equation:

$$(-2)(x-3) + (-9)(y-5) + (-1)(z-1) = 0$$

4. (2 points) Find the equation of the plane that contains the point $A(3, 4, 5)$ and the following line

$$\frac{x-3}{2} = \frac{y+1}{3} = z$$

$B(3,-1,0)$ is on the plane



the vectors

$\overrightarrow{AB} = (0,-5,-5)$

and $\vec{u} = (2,3,1)$

are both in the plane

$\vec{n} = \overrightarrow{AB} \times \vec{u}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -5 & -5 \\ 2 & 3 & 1 \end{vmatrix} = (-5-(-15))\vec{i} - (0-(-10))\vec{j} + (0-(-10))\vec{k}$$

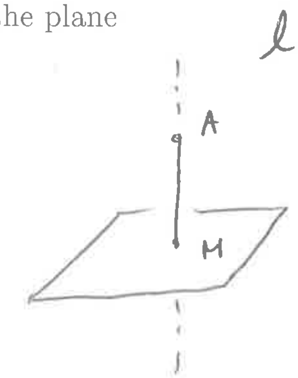
$$= (10, 10, 10)$$

the equation of the plane is

$10(x-3) + 10(y-4) + 10(z-5) = 0$

5. (2 points) Find the distance between the point $P(0, 2, 5)$ and the plane

$$2x - 3y + z = 9$$



the direction of l: $(2, -3, 1)$

the parametric equations for l,

$$x = 2\lambda + 0$$
$$y = -3\lambda + 2$$
$$z = \lambda + 5$$

M is both on l and P:

$$2(2\lambda + 0) - 3(-3\lambda + 2) + 1(\lambda + 5) = 9$$

$$4\lambda + 9\lambda - 6 + \lambda + 5 = 9$$

$$14\lambda = 10$$

$$\lambda = \frac{10}{14} = \frac{5}{7}$$

$$x_M = 2\lambda = \frac{10}{7} \qquad y_M = -3\lambda + 2 = -\frac{15}{7} + 2 = \frac{1}{7}$$

$$z_M = \frac{5}{7} + 5 = \frac{40}{7}$$

$$|\overrightarrow{PM}| = \sqrt{\left(\frac{10}{7} - 0\right)^2 + \left(\frac{1}{7} - 2\right)^2 + \left(\frac{40}{7} - 5\right)^2}$$

6. (2 points) For the following parametric curve

$$x = \sec t + t^3 \quad = f(t)$$
$$y = e^t \cos t \quad = g(t)$$

Find $\frac{d^2y}{dx^2}$ in terms of t.

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{e^t \cos t + e^t(-\sin t)}{\tan t \sec t + 3t^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\cancel{\frac{(e^t\cos t + e^t(-\sin t) + e^t(-\sin t) + e^t(-\cos t))}{(\tan t \sec t + 3t^2)^2}}$$

$$= \cancel{\frac{\frac{1}{(\tan t\sec t + 3t^2)^2} \cdot (e^t \cos t}{\frac{dx}{dt}}}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{(\tan t \sec t + 3t^2)^2}\Big(\big(e^t\cos t + e^t(-\sin t) + e^t(-\sin t) + e^t(-\cos t)\big)\cdot(\tan t\sec t + 3t^2) - (e^t\cos t + e^t(-\sin t))(\tan t \cdot \sec t \tan t + \sec^3 t + 6t)$$

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{1}{(\tan t\sec t + 3t^2)^3}\Big(\qquad \Big)$$

7. (2 points) For the following parametric curve on the interval $0 \le t \le \frac{\pi}{4}$.

$$x = \sec t + \sin t$$
$$y = \cos t$$

Find the area under the curve from $t = 0$ to $t = \frac{\pi}{4}$.

$$\text{Area} = \int_{x(0)}^{x(\frac{\pi}{4})} y\,dx \qquad dx = (\sec t \tan t + \cos t)\,dt$$

$$= \int_0^{\frac{\pi}{4}} \cos t\,(\sec t \tan t + \cos t)\,dt$$

$$= \int_0^{\frac{\pi}{4}} (\tan t + \cos^2 t)\,dt$$

$$\int_0^{\frac{\pi}{4}} \tan t\,dt = \int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\,dt \qquad u = \cos t, \quad du = -\sin t\,dt$$

$$= \int_1^{\frac{\sqrt{2}}{2}} \frac{-du}{u} = -\ln u \Big|_1^{\frac{\sqrt{2}}{2}} = -\ln\left(\frac{\sqrt{2}}{2}\right)$$

$$\int_0^{\frac{\pi}{4}} \cos^2 t\,dt = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 2\theta)\,d\theta$$

$$= \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} = \frac{\pi}{8} - \frac{1}{4}$$

$$\text{Area} = -\ln\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{8} - \frac{1}{4}$$

MATH 2433
Summer 2017
Final Exam
8/1/2017
Time Limit: 60 Minutes

Name: ______________________

This exam contains 2 pages (including this cover page) and 4 questions.

Please write down all the explanations as well as your final answer. No calculators are allowed. **Please justify any formula that has not been mentioned in class.**

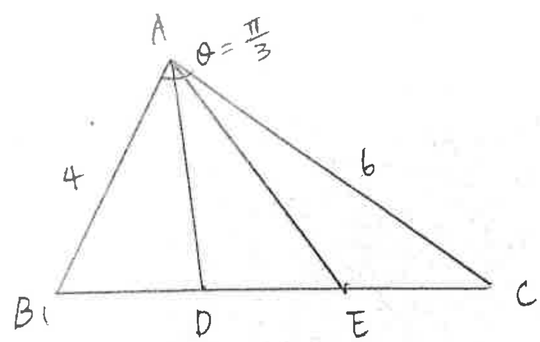
1. (5 points) Find the MacLaurin series expansion of

$$\int \ln|1+x|\,dx$$

and the interval of convergence of that series.

2. (4 points) In the triangle ABC, $|BD| = |DE| = |EC|$. $|AB| = 4$, $|AC| = 6$, the angle θ is the angle $\angle BAC$ with $\theta = \frac{\pi}{3}$.

Calculate $\overrightarrow{AD} \cdot \overrightarrow{DC}$.



3. (6 points) Find the distance between the two skew lines

$$\frac{x-1}{3} = \frac{y+2}{2} = z$$
$$x-2 = \frac{y-1}{2} = \frac{z-3}{3}$$

(Hint: there is more than one method, but you may find the distance between one line and a certain plane.)

4. (5 points) For an ellipse with major axis 20 and minor axis 10, find the largest and smallest curvature on the ellipse.
(A formula without explanation will be given little credit.)

MATH 2433 Summer 2017

Final Exam Answer key

1.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$\int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^n\,dx$$

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\int \ln|1+x|\,dx = \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{n+1}}{n+1}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{(n+1)(n+2)}$$

radius of convergence:

$$\frac{a_{n+1}}{a_n} = (-1)^{n+1} \cdot \frac{x^{n+3}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{x^{n+2}} \cdot \frac{1}{(-1)^n}$$

$$= \frac{(-1)\cdot x \cdot (n+1)(n+2)}{(n+2)(n+3)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |-1|\cdot|x|\cdot \lim_{n\to\infty} \frac{(n+1)(n+2)}{(n+2)(n+3)} = |x|$$

$$|x| < 1$$

The series is convergent on $(-1, 1)$

Endpoints:

(continued on the back)

End points: $x = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$$ is an alternating series

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(n+1)(n+2)} \right| = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = 1$$ (as a telescoping series)

therefore $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$ is absolutely convergent,

therefore convergent.

$x = -1$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$$ is convergent by the argument above

therefore the interval of convergence is $[-1, 1]$

2.

a) $\vec{AD} = \vec{AB} + \vec{BD}$

b) $\vec{AD} = \vec{AC} + \vec{CD} = \vec{AC} - \vec{DC} = \vec{AC} - 2\vec{BD}$

a) ×2 + b) :

$$3\vec{AD} = 2\vec{AB} + \vec{AC}$$

$$\vec{AD} = \frac{2}{3}\vec{AB} + \frac{1}{3}\vec{AC}$$

$$\vec{DC} = \frac{2}{3}\vec{BC} = \frac{2}{3}(\vec{AC} - \vec{AB})$$

$$\vec{AD} \cdot \vec{DC} = \left(\frac{2}{3}\vec{AB} + \frac{1}{3}\vec{AC}\right) \cdot \frac{2}{3}(\vec{AC} - \vec{AB})$$

$$= \frac{4}{9}\vec{AB} \cdot \vec{AC} + \frac{2}{9}\vec{AC} \cdot \vec{AC} - \frac{4}{9}\vec{AB} \cdot \vec{AB} - \frac{2}{9}\vec{AC} \cdot \vec{AB}$$

$$= \frac{4}{9}|\vec{AB}| \cdot |\vec{AC}| \cdot \cos\theta + \frac{2}{9}|\vec{AC}|^2$$

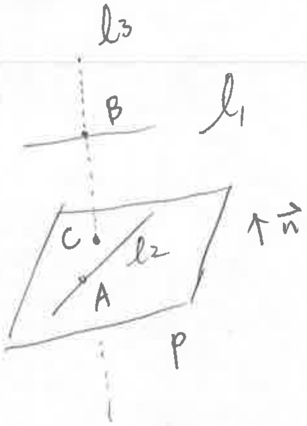
$$- \frac{4}{9}|\vec{AB}|^2 - \frac{2}{9}|\vec{AC}| \cdot |\vec{AB}| \cdot \cos\theta$$

$$= \frac{4}{9} \cdot 4 \cdot 6 \cdot \cos\frac{\pi}{3} + \frac{2}{9} \cdot 6^2 - \frac{4}{9} \cdot 4^2 - \frac{2}{9} \cdot 4 \cdot 6 \cdot \cos\frac{\pi}{3}$$

$$= 12 \cdot \frac{1}{2} + 8 - \frac{64}{9}$$

$$= 14 - \frac{64}{9} = \frac{62}{9}$$

3



the plane p ~~pas~~ that contains l_2 and parallel to l_1 has a normal vector

$$\vec{n} = (3,2,1) \times (1,2,3)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (4,-8,4)$$

A point $A(2,1,3)$ is on l_2.

equation of p:

$$4(x-2) - 8(y-1) + 4(z-3) = 0$$

A point $B(1,-2,0)$ is on l_1

A line l_3 passing through B and perpendicular to p has parametric equations

$$x = 1 + 4\lambda$$

$$y = -2 - 8\lambda$$

$$z = 4\lambda$$

let C be the ~~intercepti~~ intersection between l_3 and p

$$4(1+4\lambda-2) - 8(-2-8\lambda-1) + 4(4\lambda-3) = 0$$

$$16\lambda - 4 + 24 + 64\lambda + 16\lambda - 12 = 0$$

$$96\lambda + 8 = 0$$

$$\lambda = -\frac{1}{12}$$

③ cont.)

$$C\left(1-\frac{4}{12},\ -2+\frac{8}{12},\ -\frac{4}{12}\right)$$

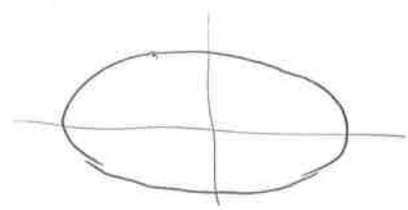
$$C\left(\frac{2}{3},\ -\frac{4}{3},\ -\frac{1}{3}\right)$$

distance between l_1 & l_2

$$|\vec{BC}| = \sqrt{\left(1-\frac{2}{3}\right)^2+\left(-2+\frac{4}{3}\right)^2+\left(0+\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{9}} = \frac{\sqrt{6}}{3}$$

④. $a = \frac{20}{2} = 10, \quad b = \frac{10}{2} = 5$



let the ellipse be given by

$$x = 10\cos t$$

$$y = 5\sin t$$

$$z = 0$$

$$\vec{r}' = (-10\sin t,\ 5\cos t,\ 0) \qquad |\vec{r}'| = \sqrt{100\sin^2 t + 25\cos^2 t}$$

$$\vec{r}'' = (-10\cos t,\ -5\sin t,\ 0)$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -10\sin t & 5\cos t & 0 \\ -10\cos t & -5\sin t & 0 \end{vmatrix} = (0,0,\ 50\sin^2 t + 50\cos^2 t)$$

$$= (0, 0, 50)$$

$$|\vec{r}' \times \vec{r}''| = 50$$

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{50}{\sqrt{100\sin^2 t + \underset{25}{\cancel{50}}\cos^2 t}^{\,3}}$$

largest curvature: right most point $t=0$

$$\kappa = \frac{50}{\sqrt{25}^{\,3}} = \frac{50}{125} = \frac{2}{5}$$

Smallest curvature: top point $t = \frac{\pi}{2}$

$$\kappa = \frac{50}{\sqrt{100}^{\,3}} = \frac{50}{1000} = \frac{1}{20}$$