

Math 2423
Spring 2016
Exam 2
3/25/16
Time Limit: 50 Minutes

Name (Print): Answer Key

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

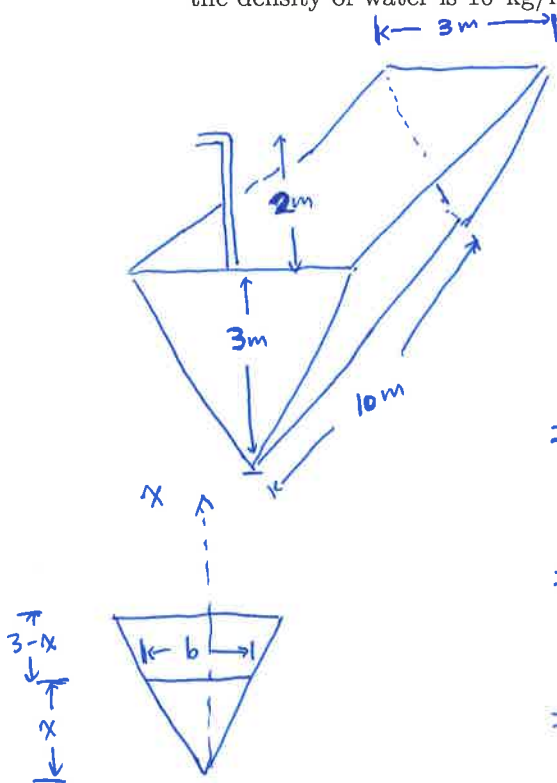
You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- Partial credits apply. Part of an important step in mathematics is to write what you think in a coherent way, even when it does not yield fruitful results.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	30	
6	20	
Total:	100	

1. (15 points) Find the work required to pump the water out of a tank through a pipe on the top of the tanks. The dimensions are shown in the picture. You may use the following constants: the density of water is 10^3 kg/m^3 , the gravity constant is 9.8 N/kg .



$$\frac{b}{x} = \frac{3}{3}$$

$$b = x \quad (4')$$

$$\int (\text{Volume of } dx\text{-layer at height } x) \cdot \text{density} \cdot (\text{gravity constant}) \cdot (\text{distance lifted}) \quad (3')$$

$$= \int_0^3 (\text{area} \cdot dx) \cdot 10^3 \cdot 9.8 \cdot ((3-x) + 2) \quad (3')$$

$$= \int_0^3 (b \cdot 10 \cdot dx) \cdot 10^3 \cdot 9.8 (5-x)$$

$$= \int_0^3 (x \cdot 10 \cdot dx) \cdot 10^3 \cdot 9.8 (5-x)$$

$$= 10^4 \cdot 9.8 \int_0^3 x(5-x) dx$$

$$= 9.8 \times 10^4 \left(\frac{5x^2}{2} - \frac{x^3}{3} \right) \Big|_0^3 \quad (3')$$

$$= 9.8 \times 10^4 \left(\frac{45}{2} - \frac{27}{3} \right)$$

$$= 9.8 \times 10^4 \left(\frac{45}{2} - 9 \right)$$

$$= 9.8 \times 10^4 \left(\frac{45}{2} - \frac{18}{2} \right)$$

$$= 9.8 \times 10^4 \cdot \frac{27}{2}$$

$$= 4.9 \times 27 \times 10^4$$

$$= 132.3 \times 10^4 \text{ J}$$

2. (10 points) Find the average of the function $y = e^x$ over the interval $[0, 1]$.

$$\frac{1}{1-0} \cdot \int_0^1 e^x dx \quad (4')$$

$$= \frac{1}{1} \cdot e^x \Big|_0^1 \quad (3')$$

$$= e^1 - e^0 \quad (2')$$

$$= e - 1 \quad (1')$$

3. (10 points) Let f be a one-to-one, continuous function for which $f'(x)$ exists on \mathbb{R} . Show that

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

$$f(f^{-1}(x)) = x \quad \text{by the property of the inverse functions} \quad (3')$$

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x \quad (3')$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1 \quad \text{by chain Rule} \quad (3')$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \quad (1)$$

4. The following questions are based on the definition $\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$.

(a) (10 points) Show that $\frac{d}{dx}\exp(x) = \exp(x)$, without using the fact that $\exp(x) = e^x$.

Differentiate the right hand side term by term,

$$\frac{d}{dx} \exp(x) = \frac{d}{dx} \left(1 + x + \frac{x^2}{2} + \dots \right) \quad (3')$$

$$= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \quad (2')$$

$$= \cancel{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}$$

$$= 0 + 1 + x + \frac{3 \cdot x^2}{3 \cdot 2 \cdot 1} + \frac{4x^3}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{5x^4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \dots \quad (3')$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (1')$$

$$= \exp(x) \quad (1')$$

(b) (5 points) Estimate e using the first four terms in the sum.

$$e = \exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} \quad (2') \quad (\text{circled } 30)$$

$$(3') \quad = 1 + 1 + \frac{1}{2} + \frac{1}{6}$$

$$= 2 + \frac{4}{6} = 2\frac{2}{3}$$

(c) (10 points) Evaluate $\int \tan x \, dx$.

$$\int \tan x \, dx$$
$$= \int \frac{\sin x}{\cos x} \, dx \quad (z')$$

$$u = \cos x \quad (z')$$
$$du = -\sin x \, dx \quad -du = \sin x \, dx \quad (z')$$

$$\int \tan x \, dx = \int \frac{-du}{u} = - \int \frac{1}{u} \, du \quad (z')$$
$$= - \ln|u| + C \quad (z')$$
$$= - \ln|\cos x| + C$$

5. (a) (10 points) Evaluate $\frac{d}{dx} x^{\sqrt{x}}$.

$$y = x^{\sqrt{x}}$$

$$\ln y = \ln(x^{\sqrt{x}}) \quad (3')$$

$$\ln y = \sqrt{x} \ln x \quad (1')$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sqrt{x} \ln x) \quad (2')$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x} \quad (3')$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}}$$

$$\frac{dy}{dx} = y \cdot \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right) = x^{\sqrt{x}} \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right) \quad (1')$$

(b) (10 points) Evaluate $\frac{d}{dx} \frac{x^2 e^x}{\sqrt{x^2+x}}$.

$$y = \frac{x^2 e^x}{\sqrt{x^2+x}}$$

$$\ln y = \ln(x^2 e^x) - \ln \sqrt{x^2+x} \quad (+3')$$

$$\ln y = \ln(x^2) + \ln(e^x) - \frac{1}{2} \ln(x^2+x) \quad (+3')$$

$$\ln y = 2 \ln x + x - \frac{1}{2} \ln(x^2+x)$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left(2 \ln x + x - \frac{1}{2} \ln(x^2+x) \right) \quad (+3')$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{x} + 1 - \frac{1}{2} \cdot \frac{2x+1}{x^2+x}$$

$$\frac{dy}{dx} = y \cdot \left(\frac{2}{x} + 1 - \frac{1}{2} \cdot \frac{2x+1}{x^2+x} \right) \quad (1')$$

$$\frac{dy}{dx} = \frac{x^2 e^x}{\sqrt{x^2+x}} \left(\frac{2}{x} + 1 - \frac{1}{2} \cdot \frac{2x+1}{x^2+x} \right)$$

(Correct usage of quotient rule will earn full credit.)

6. There was 100mg of Carbon-14 initially subject to natural decay. After 11460 years, only 25mg was left. (There might be redundancy in the numeric information given in this problem and it is okay to not use all of the numbers.)

(a) (10 points) Find a model for the mass of Carbon-14 as a function of time in years. You may use the fact that $\ln 4 = 1.38629436112$ and $\frac{1.38629436112}{11460} = 0.00012096809$.

$$25 = 100 e^{-k \cdot 11460} \quad (3')$$

$$\frac{25}{100} = e^{-k \cdot 11460}$$

$$\frac{1}{4} = e^{-k \cdot 11460} \quad (1')$$

$$\ln\left(\frac{1}{4}\right) = \ln(e^{-k \cdot 11460}) \quad (3')$$

$$\ln 1 - \ln 4 = -k \cdot 11460$$

$$0 - \ln 4 = -k \cdot 11460 \quad (2')$$

$$k = \frac{\ln 4}{11460} = 0.00012096809 \text{ yrs}^{-1} \quad (1')$$

$$y = 100 e^{-0.00012096809 t}$$

(b) (10 points) State the decay rate and half-life of Carbon-14. You may use the fact that $\ln 2 = 0.69314718056$ and $\frac{0.69314718056}{0.00012096809} = 5730$.

Decay rate: $0.00012096809 \text{ yrs}^{-1}$

half-life:

$$50 = 100 e^{-kT} \quad (2')$$

$$\cancel{50 = 100 e^{-kT}}$$

$$50 = 100 e^{-0.00012096809 T} \quad (1')$$

$$\frac{50}{100} = e^{-0.00012096809 T} \quad (1')$$

$$\ln\left(\frac{1}{2}\right) = -0.00012096809 T \quad (3')$$

$$\ln 1 - \ln 2 = -0.00012096809 T$$

$$0 - \ln 2 = -0.00012096809 T \quad (2')$$

$$T = \frac{-\ln 2}{-0.00012096809} = 5730 \text{ yrs} \quad (1')$$