1 Lecture 2/29

1.1 Definition of a function

A function $f : \mathbb{R}$(the domain)$\rightarrow$ $\mathbb{R}$(the codomain), where $\mathbb{R}$ is the collection(set) of real numbers, assigns to every number in the domain, a **unique** number in the codomain.

1.1.1 Ex. 1

$y = x^2 + 2x + 3$ is a function.

1.1.2 Ex. 2

$y = \pm\sqrt{r^2 - x^2}$ is **not** a function.

1.1.3 Ex. 3

$y = \frac{x^2+1}{x-1}$ is a function from $\{$all real numbers x such that $x \neq 1\}$ to $\mathbb{R}$.

1.2 Inverse functions

If f and g are two functions, $f \circ g$ is a function given by $f \circ g(x) = f(g(x))$.

We want to define f^{-1} such that $f^{-1} \circ f(x) = \mathrm{id}(x) = x$. ($\frac{1}{f(x)}$ would not be such a function.)

Let

$$f(b) = \text{the number } a \text{ that yields the number } b \text{ when plugged into the function } f$$

1.2.1 Ex. 4

Find the inverse function of $y = \frac{x+1}{x-1}$. we need to switch out the x's and y's and solve for y(Zainab.)

$$x = \frac{y+1}{y-1} \tag{1}$$
$$(y-1)x = y+1 \tag{2}$$
$$xy - x = y+1 \tag{3}$$
$$xy - y = x+1 \tag{4}$$
$$(x-1)y = x+1 \tag{5}$$
$$y = \frac{x+1}{x-1} \tag{6}$$

1.2.2 Ex. 5

Find the inverse function of $y = \sqrt{x+1} + \sqrt{x+3}$. The domain of this function is $[-1, \infty)$(Jeff.)

$x + 1 = 0$ yields $x = -1$, which implies $[-1, \infty)$.

$x + 3 = 0$ yields $x = -3$, which implies $[-3, \infty)$.

We can square both sides to get rid of the radical(Zainab.)

$$y^2 = (\sqrt{x+1})^2 + 2\sqrt{x+1}\sqrt{x+3} + (\sqrt{x+3})^2 \tag{7}$$
$$y^2 = x + 1 + 2\sqrt{x^2 + 4x + 3} + x + 3 \tag{8}$$
$$y^2 = 2x + 4 + 2\sqrt{x^2 + 4x + 3} \tag{9}$$
$$y^2 - (2x+4) = 2\sqrt{x^2 + 4x + 3} \tag{10}$$
$$y^4 - 2y^2(2x+4) + (2x+4)^2 = 4(x^2 + 4x + 3) \tag{11}$$
$$y^4 - 4xy^2 - 8y^2 + 4x^2 + 16x + 16 = 4x^2 + 16x + 12 \tag{12}$$
$$y^4 - 4xy^2 - 8y^2 + 16 = 12 \tag{13}$$
$$y^4 - 8y^2 + 16 - 12 = 4xy^2 \tag{14}$$
$$\frac{y^4 - 8y^2 + 4}{4y^2} = x \tag{15}$$

1.3 Derivatives of the inverse

Theorem 1.

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

1.3.1 Ex. 6

For $f(x) = x^2 + 3x + 3$, a function from {all positive real numbers} to $\mathbb{R}$, find $f^{-1}(7)$.

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} \tag{16}$$

$$7 = x^2 + 3x + 3 \tag{17}$$
$$0x^2 + 3x - 4 \tag{18}$$
$$0 = (x+4)(x-1) \tag{19}$$
$$x = -4, x = 1 \tag{20}$$

$$(f^{-1})'(7) = \frac{1}{f'(1)} \tag{21}$$
$$= \frac{1}{2 \times 1 + 3} \tag{22}$$
$$= \frac{1}{5} \tag{23}$$

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2.1 More on inverse functions

The function $y = x^2 + 3x + 3$ when we computed $(f^{-1})'(7)$ have two x values at $y = 7$.

A *one-to-one* function is a function that has a unique x-value when solving it for each y-value.

2.1.1 Ex. 1

$y = x^2$ is not one-to-one.

2.1.2 Ex. 2

$y =$sin(x) is not one-to-one:

$$1 = \sin(x)(\text{Jeff}) \tag{24}$$
$$x = \frac{\pi}{2}, \frac{5\pi}{2} \ldots \tag{25}$$

The codomain is the set of potential y-values you can get, where the range is the exact set of y-values you do get. Let $y = x^2$ be a function $\mathbb{R} \to \mathbb{R}$, the range is $[0, \infty)$.

2.2 Proof for Theorem 1

Proof. f^{-1} is designed so that

$$f^{-1} \circ f(x) = \text{id}(x) = x \tag{26}$$
$$(\text{or})f^{-1}(f(x)) = x \tag{27}$$

similarly,

$$f(f^{-1}(x)) = x \tag{28}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} f(f^{-1}(x)) = \frac{\mathrm{d}}{\mathrm{d}x} x \tag{29}$$

Using $u = f^{-1}(x)$,

$$f'(u)\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \tag{30}$$
$$f'(f^{-1}(x))(f^{-1})'(x) = 1 \tag{31}$$
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \tag{32}$$

□

2.3 The exponential function

We are trying to understand the question of:

- why is $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$
- how to compute e.

2.3.1 Definition of the function "exp"

Define

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots \tag{33}$$

as an infinite sum(Calc III stuff.) Where $3! = 3 \times 2 \times 1$ etc.

Fact.

$$\exp(x+y) = \exp(x) \cdot \exp(y) \tag{34}$$

To show this, we need to show

$$1 + (x+y) + \frac{(x+y)^2}{2} + \frac{(x+y)^3}{3!} \cdots = (1 + x + \frac{x^2}{2} + \cdots)(1 + y + \frac{y^2}{2} + \cdots) \tag{35}$$

For a specific term $\frac{x^{10}y^{12}}{10!\cdot 12!}$(or $\frac{x^m y^n}{m!\cdot n!}$ generically), we need to compare that with the combined term on the left hand side. On the left, this term comes from

$$(x+y)(x+y)\cdots(22 \text{ times})\cdots(x+y) \tag{36}$$

which is

$$\frac{(\text{the number of ways to choose 10 things out of 22 things})x^{10}y^{12}}{22!} \tag{37}$$

This is equal to $\frac{x^{10}y^{12}}{10!\cdot 12!}$ based on an additional problem in HW#5.

2.4 Further properties of "exp"

Based on another HW problem, you can show that

$$\exp(x) = (\exp(1))^x \tag{38}$$

Let $e = \exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} \cdots = 2.718$

Also,

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = 1 + \frac{2x}{2} + \frac{3x^2}{3 \cdot 2 \cdot 1} + \cdots \text{(Dhruv \& Josh L.)} \tag{39}$$

$$= e^x \tag{40}$$

Note: e^x is not a **power** of x, therefore we cannot apply the power rule. In other words, **exponential functions**(things like 3^x) are different objects from **polynomials** or **powers** of x(things like x^3.)

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3.0.1 Ex.

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \tag{41}$$

3.0.2 Ex.

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{x^2}) = e^{x^2}2x \tag{42}$$

3.0.3 Ex.

$$\int e^{2x}\,\mathrm{d}x = \begin{cases} 2e^{2x}? \\ \frac{1}{2}e^{2x}? \end{cases} \tag{43}$$

($u = 2x$(Whitney,) $\mathrm{d}u = 2\mathrm{d}x$, $\frac{1}{2}\mathrm{d}u = \mathrm{d}x$):

$$= \int e^u (\frac{1}{2}\mathrm{d}u) \tag{44}$$

$$= \frac{1}{2}\int e^u \mathrm{d}u \tag{45}$$

$$= \frac{1}{2}e^u + C \tag{46}$$

$$= \frac{1}{2}e^{2x} + C \tag{47}$$

3.0.4 Ex.

$$\int \frac{e^x - 1}{e^{2x}}\,\mathrm{d}x = \int (e^x - 1)(e^{-2x}) \tag{Braden}$$

$$= \int e^x e^{-2x} - e^{-2x}\,\mathrm{d}x \tag{48}$$

$$= \int (e^{-x} - e^{-2x})\mathrm{d}x \tag{49}$$

$$= -e^x + \frac{1}{2}e^{-2x} + C \tag{Whitney}$$

3.0.5 Ex.

$$\int e^{2x^2} x\,\mathrm{d}x \tag{50}$$

$$(u = 2x^2, \mathrm{d}u = 4x\mathrm{d}x, \frac{1}{4}\mathrm{d}u = x\mathrm{d}x) \tag{51}$$

$$= \int e^u (\frac{1}{4}\mathrm{d}u) \tag{52}$$

$$= \frac{1}{4}e^u + C \tag{53}$$

$$= \frac{1}{4}e^{2x^2} + C \tag{54}$$

3.1 Properties of exp

$$e^x e^y = e^{x+y} \qquad \text{(Dhruv: we proved this yesterday)}$$
$$\frac{e^x}{e^y} = e^{x-y} \qquad \text{(Kennedy)}$$
$$(e^x)^y = e^{xy} \qquad (55)$$

proof of (55).

$$(e \cdot e \cdots x \text{ copies} \cdots e) \cdots y \text{ blocks} \cdots (e \cdot e \cdots x \text{ copies} \cdots e) \qquad (56)$$
$$= e \cdot e \cdots xy \text{ copies} \cdots e \qquad \text{(Jeff)}$$

□

3.2 The natural logarithmic function

We want to study the inverse function $\exp^{-1}$:

$$\exp^{-1}(b) = \text{the number } a \text{ such that } \exp(a) = b \qquad (57)$$

We call $\exp^{-1}$ the **natural logarithmic** function, and denote it as $\ln(x)$.

3.2.1

$$\ln(10) = ? \qquad (58)$$
$$\ln(e^5) = 5 \qquad (59)$$
$$e^{\ln(3)} = 3 \qquad (60)$$

3.2.2

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x} \qquad (61)$$

Proof.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \tag{Jeff}$$

$$\ln'(x) = \frac{1}{\exp'(\ln(x)} \tag{62}$$

$$= \frac{1}{\exp(\ln(x)} \tag{Braden}$$

$$= \frac{1}{x} \tag{63}$$

□

3.3 Properties of ln

$$\ln(x) + \ln(y) = \ln(xy) \tag{Kennedy}$$

$$\ln(x) - \ln(y) = \ln(\frac{x}{y}) \tag{64}$$

$$\ln(x \cdot x \cdots y \text{ times} \cdots x) = \tag{65}$$

$$\ln(x) + \ln(x) + y \text{ times} \cdots \ln(x) \tag{66}$$

$$\ln(x^y) = y\ln(x) \tag{67}$$

3.3.1 Ex.

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\frac{x^2+1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x^2+1) - \ln(x)) \tag{Dhruv}$$

$$= \frac{1}{x^2+1}(2x) - \frac{1}{x} \tag{68}$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} \tag{69}$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| \tag{70}$$

Negative numbers are not in the domain of $\ln x$.

4.0.1 Ex.

$$\int \tan x \, \mathrm{d}x \tag{71}$$

$$\text{Trial: } u = \tan x, \mathrm{d}u = \sec^2 x \, \mathrm{d}x, \frac{1}{\sec^2 x}\mathrm{d}u = \mathrm{d}x \tag{72}$$

$$\int \frac{u \, \mathrm{d}u}{\sec^2 x} = \int \frac{u}{\frac{1}{(\cos x)^2}}\mathrm{d}u \tag{73}$$

$$= \int u(\cos x)^2 \, \mathrm{d}u \tag{74}$$

$$\int \frac{\sin x}{\cos x}\mathrm{d}x \tag{Jeff}$$

$$(u = \cos x, \mathrm{d}u = -\sin \mathrm{d}x) \tag{75}$$

$$= \int \frac{-\mathrm{d}u}{u} \tag{76}$$

$$= -\ln|u| + C \tag{77}$$

$$= -\ln|\cos x| + C \tag{78}$$

4.0.2 Ex.

$$\int \frac{\ln x}{x}\mathrm{d}x \tag{79}$$

$$(u = \ln(x), \mathrm{d}u = \frac{1}{x}\mathrm{d}x) \tag{80}$$

$$= \int u \, \mathrm{d}u = \frac{1}{2}u^2 + C \tag{81}$$

$$= \frac{1}{2}(\ln x)^2 + C \tag{82}$$

4.0.3 Ex.

$$\frac{\mathrm{d}}{\mathrm{d}x}10^x \tag{83}$$

$$\ln(e^{10}) = e^{\ln(10)} = 10 \tag{84}$$

$$10^x = (e^{\ln(10)})^x \tag{85}$$
$$((5^6)^7) = 5^{42}) \tag{86}$$
$$= e^{x\ln 10} = e^{(\ln 10)x} \tag{Zainab}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}10^x = \frac{\mathrm{d}}{\mathrm{d}x}(e^{10\ln x}) \tag{87}$$
$$= e^{10\ln x}(x\ln 10)' \tag{Jeff}$$
$$= e^{10\ln x}\ln 10 \tag{88}$$
$$= 10^x\ln 10 \tag{89}$$

4.0.4 Ex

$$\int 10^x \, \mathrm{d}x \tag{90}$$
$$= \int (e^{x\ln 10}) \tag{Robert. S}$$
$$(u = x\ln 10, \mathrm{d}u = (\ln 10)\mathrm{d}x) \tag{91}$$
$$= \int \frac{e^u}{\ln 10}\mathrm{d}u \tag{92}$$
$$= \frac{e^u}{\ln 10} + C \tag{93}$$
$$= \frac{e^{x\ln 10}}{\ln 10} + C \tag{94}$$
$$= \frac{10^x}{\ln 10} + C \tag{Hosain}$$

4.1 Domain, Range, Graph of e^x

$$e^x \tag{95}$$
$$e^{-100} = \frac{1}{e^{100}} \tag{96}$$

Domain: $\mathbb{R}$, Range: all positive real numbers

$$\ln x = \text{The number } y \text{ such that } e^y = x \tag{97}$$

Domain: all positive real numbers, Range: $\mathbb{R}$

$$y = e^x(+) \tag{98}$$
$$y' = e^x(+) \tag{99}$$
$$y'' = e^x(+) \tag{100}$$

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5.1 The Graph of e^x

$$\lim_{x\to\infty} e^x = \infty \tag{101}$$
$$\lim_{x\to-\infty} e^x \tag{102}$$
$$= \lim_{a\to\infty} e^{-a} \tag{103}$$
$$= \lim_{a\to\infty} \frac{1}{e^a} = 0 \tag{Braden, Kobie, Dhruv}$$

$e^0 = 1$.

$$\lim_{x\to 0^+} \ln x = -\infty(\text{or}1?) \tag{104}$$
$$\lim_{x\to 1} \ln x = 0 \tag{105}$$
$$\lim_{x\to\infty} \ln x = \infty \tag{106}$$

5.2

$$\frac{\mathrm{d}}{\mathrm{d}x}10^x = (\ln 10)10^x \tag{107}$$
$$\int 10^x \,\mathrm{d}x = \frac{10^x}{\ln 10} + C \tag{108}$$

5.2.1 Ex.

$$\frac{\mathrm{d}}{\mathrm{d}x}x^x = (\ln x)x^x, \frac{x^x}{\ln x}? \tag{109}$$
$$\text{or does it only work for} \frac{a^x}{b}? \tag{110}$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}((e^{\ln x})^x) \tag{111}$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}(e^{x\ln x}) \tag{Braden}$$
$$= e^{x\ln x}(x\ln x)' \tag{Jeff}$$
$$= e^{x\ln x}(1 \cdot \ln x + x \cdot \frac{1}{x}) \tag{112}$$
$$= e^{x\ln x}(\ln x + 1) \tag{113}$$
$$= x^x \ln x + x^x \tag{114}$$

5.3

$$\text{Rule \#1 } \ln(xy) = \ln x + \ln y \tag{115}$$
$$\text{Rule \#2 } \ln(\frac{x}{y}) = \ln x - \ln y \tag{116}$$
$$\text{Rule \#3 } \ln(x^y) = y\ln x \tag{117}$$
$$\text{Rule \#4 } \log_{10} x = \frac{\ln x}{\ln 10} \tag{118}$$

5.4

I want to study the inverse function of $y = f(x) = 10^x$.

$$f^{-1}(x) = \text{the number } y \text{ such that } 10^y = x \tag{119}$$

For example,

$$f^{-1}(10) = 1 \tag{Braden}$$
$$f^{-1}(100) = 2 \tag{Kennedy}$$
$$\vdots \tag{120}$$

We call $f^{-1}(x)$ as $\log_{10} x$.

5.4.1 Ex.

$$\frac{\mathrm{d}}{\mathrm{dx}}\frac{(x^2+2x)\sqrt{\tan x}\ \ln x}{x^2\sin xe^x} \tag{121}$$

$$y = \frac{(x^2+2x)\sqrt{\tan x}\ln x}{x^2\sin xe^x} \tag{122}$$

$$\ln(y) = \ln((x^2+2x)\sqrt{\tan x}\ln x) - \ln(x^2\sin xe^x) \tag{123}$$

$$= \ln(x^2+2x) + \ln(\sqrt{\tan x}) + \ln(\ln x) \tag{124}$$

$$- (\ln(x^2) + \ln(\sin x) + \ln(e^x)) \tag{125}$$

$$= \ln(x(x+2)) + \ln(\sqrt{\tan x}) + \ln(\ln x) \tag{126}$$

$$- (\ln(x^2) + \ln(\sin x) + \ln(e^x)) \tag{Braden}$$

$$= \ln(x) + \ln(x+2) + \frac{1}{2}\ln(\tan x) + \ln(\ln x) \tag{127}$$

$$- 2\ln(x) - \ln(\sin x) - x \tag{Josh L.}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}(\ln y)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} + \frac{1}{x+2} + \frac{1}{2}\frac{\sec^2 x}{\tan x} + \frac{1}{\ln x}\frac{1}{x} - \frac{2}{x} - \frac{\cos x}{\sin x} - 1 \tag{128}$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} + \frac{1}{x+2} + \frac{1}{2}\frac{\sec^2 x}{\tan x} + \frac{1}{\ln x}\frac{1}{x} - \frac{2}{x} - \frac{\cos x}{\sin x} - 1 \tag{129}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(\frac{1}{x} + \frac{1}{x+2} + \frac{1}{2}\frac{\sec^2 x}{\tan x} + \frac{1}{\ln x}\frac{1}{x} - \frac{2}{x} - \frac{\cos x}{\sin x} - 1) \tag{130}$$

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$$y = e^t \tag{131}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^t \tag{132}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ke^t \tag{133}$$

$$k : \text{constant} \tag{134}$$

$$\text{guess:} y = Cte^t? \tag{135}$$

$$= \text{(or) } Ce^t \tag{136}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ce^t = kce^t \tag{137}$$

$$k = 1 \tag{138}$$

6.1

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln(y)) = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{Jeff}$$
$$= \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{139}$$
$$= \frac{1}{y}ky \tag{140}$$
$$= k \tag{141}$$

$$\int \frac{\mathrm{d}}{\mathrm{d}t}(\ln(y))\,\mathrm{d}t = \int k\,\mathrm{d}t \tag{142}$$
$$\ln(y) = kt + C \tag{Braden}$$
$$y = e^{kt+C} \tag{143}$$
$$y = e^{kt}e^{C} \tag{144}$$
$$y = C'e^{kt} \tag{145}$$

6.2

6.2.1 Ex.

Let y be the # of population at time t, k_1 be the rate of newborns, k_2 be the mortality rate.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k_1 y - k_2 y \tag{146}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = (k_1 - k_2)y \tag{147}$$
$$y = C'e^{kt} \tag{148}$$

Where $k = k_1 - k_2$ is the growth rate.

The world population is 2560 millions in the year of 1950 ($t = 0$) (Kennedy, Josh), 3040 millions in the year of 1960 ($t = 10$).
1) Find the growth rate of the world population.
2) Estimate the world population in the year of 2020 (t=70).

$$2560 = C'e^{k\cdot 0} \tag{149}$$
$$2560 = C' \tag{150}$$

$$3040 = 2560e^{k\cdot 10} \tag{Kennedy}$$
$$\frac{3040}{2560} = e^{k\cdot 10} \tag{151}$$
$$\ln(\frac{3040}{2560}) = 10k \tag{152}$$
$$k = 0.017185 \tag{153}$$

$$y = 2560e^{0.017185\cdot 70} \tag{Kennedy}$$
$$= 8524 \tag{154}$$

6.2.2 Ex.

Let y be the mass of a radioactive matter.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky \tag{155}$$
$$\text{grams/yrs} = \frac{1}{\text{yrs}} \cdot \text{grams} \tag{Kennedy}$$

The half life of a radioactive matter is the time it take for it to decay to half its mass.

The half life of radium-226 is 1590 years. How long does it take for it to decay to the state when only $\frac{1}{3}$ is left?

$$33.33 = 100e^{-1590t} \tag{156}$$
$$\text{or } = 100e^{-1590k}? \tag{157}$$
$$33.33 = 100e^{-\frac{t}{1590}} \tag{158}$$
$$33.33 = 100e^{-\frac{1590}{1590}} \tag{159}$$

$$
\begin{aligned}
y &= 100e^{\frac{t}{2\cdot 1590}} && (160)\\
50 &= 100e^{\frac{1590}{2\cdot 1590}} && (161)\\
\frac{1}{2} &= e^{-\frac{1}{2}} && (162)\\
&= \frac{1}{\sqrt{2.718}} && (163)
\end{aligned}
$$

$$
\begin{aligned}
50 &= 100e^{-k1590} && \text{(Jeff)}\\
\frac{1}{2} &= e^{-k1590} && (164)\\
\ln(\frac{1}{2}) &= -1590k && (165)\\
k &= -0.00043594 && (166)
\end{aligned}
$$

$$
\begin{aligned}
33.33 &= 100e^{-0.00043594t} && (167)\\
t &= 2520 && (168)
\end{aligned}
$$

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$$y = \sin x \tag{169}$$

is one-to-one on $[-\frac{\pi}{2}, -\frac{\pi}{2}]$(Josh, L.)

Define the inverse of $\sin x$ on $[-\frac{\pi}{2}, -\frac{\pi}{2}]$ as:

$$
\begin{aligned}
\sin^{-1}x &= \text{the angle } \theta \text{ between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2} \text{ such that} && (170)\\
\sin\theta &= x && \text{(Kobie)}
\end{aligned}
$$

7.0.1 Examples

$$
\begin{aligned}
\sin^{-1}(1) &= \frac{\pi}{2} && (171)\\
\sin^{-1}(-1) &= -\frac{\pi}{2} && (172)\\
\sin^{-1}(-\frac{1}{2}) &= -\frac{\pi}{6} && \text{(Kennedy)}
\end{aligned}
$$

7.1

$$\frac{\mathrm{d}}{\mathrm{d\,x}} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \tag{173}$$
$$\frac{\mathrm{d}}{\mathrm{d\,x}} \sin^{-1} x = \frac{1}{\sin'(\sin^{-1}(x))} \tag{174}$$
$$\frac{\mathrm{d}}{\mathrm{d\,x}} \sin^{-1} x = \frac{1}{\cos(\sin^{-1}(x))} \tag{175}$$

Let

$$\theta = \sin^{-1} x \tag{176}$$
$$\sin\theta = x \tag{Jessica}$$
$$\frac{\mathrm{d}}{\mathrm{d\,x}} \sin^{-1} x = \frac{1}{\cos\theta} \tag{177}$$

$$\sin^2\theta + \cos^2\theta = 1 \tag{Dhruv}$$
$$x^2 + \cos^2\theta = 1 \tag{178}$$
$$\cos^x = \pm\sqrt{1 - x^2} \tag{Kennedy}$$

Claim that

$$\frac{\mathrm{d}}{\mathrm{d\,x}} \sin^{-1} x = \frac{1}{\cos\theta} \tag{179}$$
$$= \frac{1}{\sqrt{1 - x^2}} \tag{180}$$

7.2

Restrict $\cos x$ to $[0, \pi]$.

7.2.1 Examples

$$\cos^{-1}(0) = \frac{\pi}{2} \tag{Kennedy}$$

7.3

$$\frac{\mathrm{d}}{\mathrm{d\,x}}\cos^{-1}x = \frac{1}{\cos'(\cos^{-1}(x))} \tag{181}$$

$$\frac{\mathrm{d}}{\mathrm{d\,x}}\cos^{-1}x = \frac{1}{-\sin(\cos^{-1}(x))} \tag{182}$$

Let

$$\theta = \cos^{-1}x \tag{183}$$

$$\cos\theta = x \tag{184}$$

$$\frac{\mathrm{d}}{\mathrm{d\,x}}\cos^{-1}x = \frac{1}{-\sin\theta} \tag{185}$$

$$= -\frac{1}{\sqrt{1-x^2}} \tag{186}$$

7.4 conclusion

sin x is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and

$$\frac{\mathrm{d}}{\mathrm{dx}}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \tag{187}$$

cos x is restricted to $[0, \pi]$, and

$$\frac{\mathrm{d}}{\mathrm{dx}}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \tag{188}$$

7.5

7.5.1 Example

$$\int \frac{1}{\sqrt{1-4x^2}}\mathrm{dx} \tag{189}$$

Let

$$u = 2x \quad \text{(Kennedy, Josh)}$$

$$\mathrm{dx} = \frac{1}{2}\mathrm{du} \tag{190}$$

$$\int \frac{1}{\sqrt{1-4x^2}}\mathrm{dx} = \frac{1}{\sqrt{1-u^2}}\frac{1}{2}\mathrm{du} \tag{191}$$

$$= \frac{1}{2}\sin^{-1}(u) + C \tag{192}$$

$$= \frac{1}{2}\sin^{-1}(2x) + C \tag{193}$$