EXAM 1
Math 2423
2-18-10

Name *key*

Row ______

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(8 points)* Give the definition of the definite integral $\int_a^b f(x)\,dx$ as a limit of sums. Remember to explain the meaning of the symbols you use.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

1 1 2 1 1

where: n is the number of subintervals $[a,b]$ is split into
2 Δx is the width of each subinterval
x_i are the sample points in each subinterval

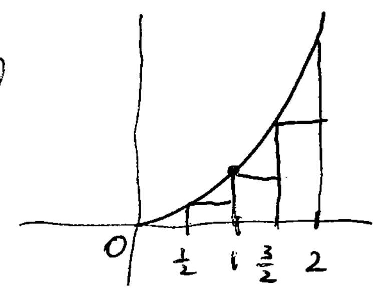
2. *(10 points)* Evaluate the Riemann sum for $f(x) = x^2$, $0 \le x \le 2$, with four subintervals, taking the sample points to be left endpoints.

$(\Delta x = \frac{1}{2})$ 2

$$R_4 = f(0)\cdot\tfrac{1}{2} + f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2}$$ 4

$$= 0^2\cdot\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2\cdot\tfrac{1}{2} + 1^2\cdot\tfrac{1}{2} + \left(\tfrac{3}{2}\right)^2\cdot\tfrac{1}{2}$$ 4

$$= 0 + \tfrac{1}{8} + \tfrac{1}{2} + \tfrac{9}{8} = \boxed{7/4}$$



3. *(12 points)* Find $\dfrac{d}{dx}\displaystyle\int_0^{\cos x} \sin(t^3)\,dt$.

Let $F(x) = \int_0^x \sin(t^3)\,dt$. 2

Then by FTC part 1, $F'(x) = \sin(x^3)$. 2

By the Chain Rule, $\frac{d}{dx}\left[\int_0^{\cos x} \sin(t^3)\,dt\right] = \frac{d}{dx}\left[F(\cos x)\right] =$ 2

$$= F'(\cos x)\cdot\frac{d}{dx}(\cos x) = \boxed{\sin\left((\cos x)^3\right)\cdot(-\sin x).}$$

2 2 2

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4. *(24 points)* Find the indefinite integral, showing all work. Remember to express your answer as a function of x.

[12] a) $\int x^2\sqrt{x+1}\,dx$

$$\boxed{\begin{aligned} \text{Let } u &= x+1 \quad ① \\ du &= dx \quad ① \\ u-1 &= x \quad ① \end{aligned}}$$

$$= \int (u-1)^2 \sqrt{u}\,du = \int (u^2-2u+1)\cdot u^{\frac{1}{2}}\,du$$ ① ① ① ①

$$\overset{②}{=} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$$

$$\overset{②}{=} \frac{2}{7}u^{7/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C}$$ ①

[12] b) $\int \frac{\tan x}{\sec^{11} x}\,dx$

$$\boxed{\begin{aligned} \text{Let } u &= \sec x \quad ① \\ du &= \sec x \tan x\,dx \quad ① \\ dx &= \frac{du}{\sec x \tan x} \quad ① \end{aligned}}$$

$$= \int \frac{\tan x}{\sec^{11} x}\cdot\frac{du}{\sec x \tan x} = \int \frac{du}{\sec^{12} x} =$$ ① ②

$$= \int \frac{du}{u^{12}} = \int u^{-12}\,du$$ ② ①

$$= \frac{u^{-11}}{-11} + C$$ ②

$$= \boxed{\frac{(\sec x)^{-11}}{-11} + C}$$ ①

$$\left(= \frac{-1}{\cos^{11} x} + C \right)$$

5. *(24 points)* Find the value of the definite integral, showing all work.

[12] a) $\int_0^1 \frac{7x^2}{\sqrt{x^3+1}}\,dx$

$$\boxed{\begin{aligned} \text{Let } u &= x^3+1 \quad ① \\ ①\ du &= 3x^2\,dx \\ ①\ dx &= \frac{du}{3x^2} \end{aligned}} \rightarrow \boxed{\begin{aligned} x=0 &\rightarrow u=1 \quad ① \\ x=1 &\rightarrow u=2 \quad ① \end{aligned}}$$

$$= \int_{u=1}^{u=2} \frac{7x^2}{\sqrt{u}}\,\frac{du}{3x^2} = \frac{7}{3}\int_{u=1}^{u=2} \frac{du}{\sqrt{u}} =$$ ① ① ①

$$= \frac{7}{3}\int_{u=1}^{u=2} u^{-\frac{1}{2}}\,du$$ ①

$$= \frac{7}{3}\cdot\left[2u^{1/2}\right]_{u=1}^{u=2}$$ ②

$$= \boxed{\frac{14}{3}(\sqrt{2}-1)}$$ ①

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[12] b) $\int_0^{\pi/4} \tan^{10} x \sec^2 x \, dx$ (simplify your answer) $= \int_{u=0}^{u=1} u^{10} \, du$ (2) $= \left[\frac{u^{11}}{11}\right]_{u=0}^{u=1}$ (2) $=$

$\boxed{\text{Let } u = \tan x \ (2), \quad du = \sec^2 x \, dx \ (2)}$ $\rightarrow$ $x = 0 \rightarrow u = \tan(0) = 0$, $x = \frac{\pi}{4} \rightarrow u = \tan\frac{\pi}{4} = 1$ (2)

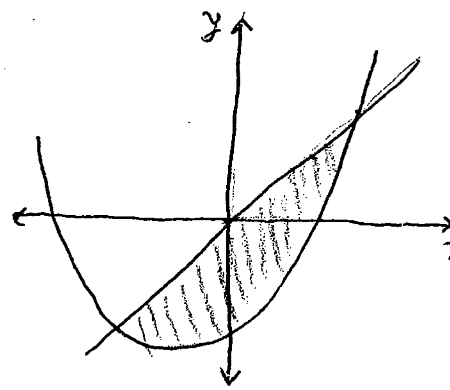
$= \frac{1}{11} - 0$

$= \boxed{\frac{1}{11}}$ (2)

6. *(22 points)* Find the areas of each of the two shaded regions indicated below.

a) The region between the graphs of $y = x$ and $y = x^2 + x - 1$

[11] Points of intersection: $x = x^2 + x - 1$
$\rightarrow 0 = x^2 - 1$
$\rightarrow x = 1$ or $x = -1$ (2)



$$A = \overset{(2)}{\int_{-1}^{1}} \left(x - \overset{(2)}{(x^2 + x - 1)}\right) dx$$

$$= \int_{-1}^{1} \left(x - x^2 - x + 1\right) dx \ (1) = \int_{-1}^{1} (1 - x^2)\, dx \ (1) = 2\int_0^1 (1 - x^2)\, dx =$$

(because $1 - x^2$ is even)

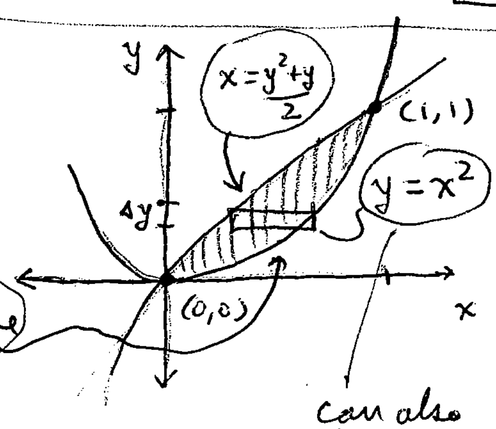
$$= 2\left[x - \frac{x^3}{3}\right]_0^1 \ (2)\ (1)$$

$$= 2\left[1 - \frac{1}{3}\right] = \boxed{4/3}.$$

b) The region between the graphs of $x = \dfrac{y^2 + y}{2}$ and $y = x^2$

[11]
$$A = \overset{(2)}{\int_{y=0}^{y=1}} \left(\overset{(2)}{\sqrt{y}} - \overset{(2)}{\left(\frac{y^2 + y}{2}\right)}\right) dy$$

(because the area of the rectangle shown here is $\left[\sqrt{y} - \left(\frac{y^2+y}{2}\right)\right] \cdot \Delta y$)



can also be written as $x = \sqrt{y}$

$$= \int_0^1 \left(y^{\frac{1}{2}} - \frac{y^2}{2} - \frac{y}{2}\right) dy \ (2)$$

$$= \left[\frac{2}{3} y^{3/2} - \frac{y^3}{6} - \frac{y^2}{4}\right]_0^1 \ (2)$$

$$= \frac{2}{3} - \frac{1}{6} - \frac{1}{4} = \boxed{\frac{1}{4}}. \ (1)$$