

Complex Analysis

Some more sample problems from the text by Brown and Churchill. As on the last sheet of sample problems I handed out, these problems are more straightforward than most of the ones in Greene and Krantz' text. Some should be quite easy, some might not be so easy. They aren't intended to cover all the topics we've discussed in class, but they might provide some useful practice.

1. Show that if $\sum_{n=1}^{\infty} z_n = z$, then $\sum_{n=1}^{\infty} \bar{z}_n = \bar{z}$.
2. Obtain power series expansions about $z = 0$ for $z \cosh(z^2)$ and $z/(z^4 + 9)$, and a power series expansion about $z = 1$ for e^z .
3. Show that if $f(z) = \sin(z^2)$ then $f^{(4n)} = 0$ and $f^{(2n+1)} = 0$ for $n = 0, 1, 2, \dots$. Here $f^{(k)}$ denotes the k th derivative of f .
4. Find the Laurent series for $1/(4z - z^2)$ in $\{0 < |z| < 4\}$.
5. Find two Laurent series for $(z + 1)/(z - 1)$ in powers of z , and specify the region where each one is valid.
6. Prove that if $f(z)$ is defined by $f(z) = \frac{\cos z}{z^2 - (\pi/2)^2}$ for $z \neq \pm\pi/2$, and $f(z) = -1/\pi$ for $z = \pm\pi/2$, then f is an entire function.
7. Use the residue theorem to evaluate the integral of each of these functions around the circle $|z| = 3$ in the counterclockwise sense: (a) $\exp(-z)/z^2$, (b) $\exp(-z)/(z - 1)^2$, (c) $z^2 \exp(1/z)$, (d) $(z + 1)/(z^2 - 2z)$. Answers: (a) $-2\pi i$, (b) $-2\pi i/e$, (c) $\pi i/3$, (d) $2\pi i$.

8. Show that if C is the circle $|z| = 1$, traversed counterclockwise, then

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

(Hint: write the integrand as $(\exp z)(\exp(1/z))$ and expand the first factor in power series. You should justify the step in which the integral of the sum becomes the sum of the integrals.)

9. Evaluate the integral $\int_C \frac{\cosh \pi z}{z(z^2 + 1)} dz$, where C is the circle $|z| = 2$, traversed counterclockwise. (Recall that $\cosh z$ is defined as $(e^z + e^{-z})/2$.) Answer: $4\pi i$.

10. Suppose p and q are holomorphic in a neighborhood of z_0 , and $p(z_0) \neq 0$. Show that if $p(z)/q(z)$ has a pole of order m at z_0 , then z_0 is a zero of order m of q . (This means that $q(z) = (z - z_0)^m h(z)$ where h is holomorphic in a neighborhood of z_0 and $h(z_0) \neq 0$.)

NOTE: Problems like the two below won't be on the second exam, because that exam only covers the first five sections of Chapter 4 (along with all of Chapter 3).

11. Use residues to show that

$$\int_0^{\infty} \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2} = \frac{\pi}{200}.$$

12. Use residues to show that if $a > 0$ and $b > 0$ then

$$\int_0^{\infty} \frac{\cos ax dx}{(x^2 + b^2)^2} = \frac{\pi}{4b^3}(1 + ab)e^{-ab}.$$