Complex Analysis I
Final Exam

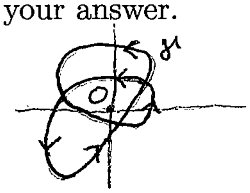
1. Suppose that $u(x,y)$ is a function defined on $D(0,1)$ such that u is C^2 on $D(0,1)$ and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for all $(x,y) \in D(0,1)$. Prove that u is C^∞ on $D(0,1)$. You may use any result from class, but be clear about which results you are using.

2. Let $f(z) = \dfrac{e^z - e^{-z}}{z^4}$.

a. Give a Laurent series for $f(z)$ in powers of z.

b. For which values of z does the Laurent series converge? Justify your answer.

c. Find $\int_\gamma f(z)\, dz$, where γ is the curve shown in the diagram:



3. Find $\displaystyle\int_C \frac{1}{z^2(z+1)^2}\, dz$, where C is the circle $\{|z| = 1/2\}$, with positive orientation. Justify your answer.

4. Suppose that $U \subseteq \mathbf{C}$ is open and connected, and f and g are holomorphic functions on U such that, for all $z \in U$, $f(z)g(z) = 0$. Show that either $f(z) = 0$ for all $z \in U$, or $g(z) = 0$ for all $z \in U$.

5. Suppose that $f(z)$ and $g(z)$ are holomorphic functions on $D(z_0, r)$, and $f(z_0) \neq 0$. Suppose $f(z)/g(z)$ has a pole of order k at z_0. Show that $g(z)$ has a zero of order k at z_0.

6. Use residues to find the value of $\displaystyle\int_0^\infty \frac{x^2\, dx}{(x^2+1)(x^2+9)}$.

7. Suppose U is an open connected subset of $\mathbf{C}$ and $S \subset U$ is discrete: that is, for every $z \in S$ there exists $\epsilon > 0$ such that there are no points of S in $D(z, \epsilon)$ besides z. Suppose f is holomorphic on $U \backslash S$ and f has a pole at every point of S. (You may assume $S \neq \emptyset$.)

a. Show that $1/f$ has a removable singularity at every point of S.

b. Show that if $w \in U$ and $f(w) = 0$, then $1/f$ has a pole at w.

c. Show that S has no accumulation points in U.

8. Suppose $p(z)$ is a polynomial of degree n, and $R > 0$ is such that $p(z) \neq 0$ for $|z| \geq R$. Find $\displaystyle\int_{|z|=R} \frac{p'(z)}{p(z)}\, dz$.

① We have that u is harmonic on $D(0,1)$, so from a result in class we know there exists a holomorphic function f on $D(0,1)$ such that $u = \operatorname{Re} f$ on $D(0,1)$.
We also know that every holomorphic function is C^∞ on its domain, so f, and therefore also u, are ~~[illegible]~~ C^∞ on $D(0,1)$.

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② ⓐ $\begin{cases} e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots \\ e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4} - \dots \end{cases}$, so $e^z - e^{-z} = 2\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)$

and $\frac{e^z - e^{-z}}{z^4} = \frac{2}{z^3} + \frac{2}{3!}\cdot\frac{1}{z} + \frac{2}{5!}z + \frac{2}{7!}z^3 + \dots$

ⓑ The function $f(z) = \frac{e^z - e^{-z}}{z^4}$ is holomorphic on $\mathbb{C}\setminus\{0\}$, so its Laurent series in powers of z converges on $\mathbb{C}\setminus\{0\}$.

ⓒ $\int_\gamma f(z)\,dz = \operatorname{Res}_f(0)\cdot \operatorname{Ind}_\gamma(0) = \frac{2}{3!}\cdot 2 = \frac{2}{3}$.

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③ The only pole of $f = \frac{1}{z^2(z+1)^2}$ within C is at $z = 0$, and since $g(z) = \frac{1}{(z+1)^2}$ has $g(0) = 1$ and $g'(0) = \frac{-2}{(z+1)^3} = -2$, then $g(z) = \sum_{k=0}^\infty a_k z^k$ where $a_0 = 1$ and $a_1 = -2$. Therefore $f(z)$ has Laurent series $\frac{1}{z^2}\left(\sum_{k=0}^\infty a_k z^k\right) = \frac{a_0}{z^2} + \frac{a_1}{z} + a_2 + a_3 z + \dots$ where $a_0 = 1$ and $a_1 = -2$, so $\operatorname{Res}_f(0) = -2$. Therefore, by the Residue Theorem, $\int_C f(z)\,dz = 2\pi i(-2) = -4\pi i$.

(4) Suppose $f \not\equiv 0$ on $\mathcal{U}$, we'll show $g \equiv 0$ on $\mathcal{U}$. Since $f \not\equiv 0$, then there exists $z_0 \in \mathcal{U}$ such that $f(z_0) \neq 0$. By continuity of f and openness of $\mathcal{U}$, there exists $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subseteq \mathcal{U}$ and $f(z) \neq 0$ for all $z \in D(z_0, \varepsilon)$. Since $f(z) g(z) = 0$ for all $z \in \mathcal{U}$, it follows that $g(z) = 0$ for all $z \in D(z_0, \varepsilon)$. Since $\mathcal{U}$ is connected, it then follows from a result in class that $g \equiv 0$ on $\mathcal{U}$.

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(5) We know $\dfrac{f(z)}{g(z)} = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$ where $a_{-k} \neq 0$, so $\forall z \in D(z_0, \varepsilon) \setminus \{z_0\}$,

$$\frac{f(z)}{g(z)} = (z - z_0)^{-k} h(z) \quad \text{where} \quad h(z) = \sum_{n=0}^{\infty} a_{n-k} (z - z_0)^n \text{ is holomorphic}$$

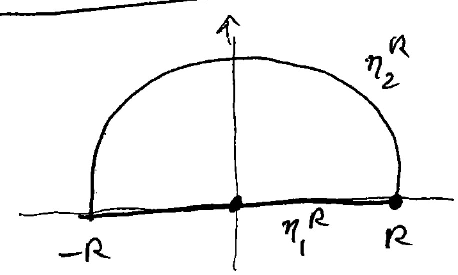
on $D(z_0, \varepsilon)$ and $h(z_0) \neq 0$. Then $g(z) = (z - z_0)^k \dfrac{f(z)}{h(z)}$, and

since $h(z_0) \neq 0$, then $\dfrac{f(z)}{h(z)}$ is holomorphic on a neighborhood of z_0,

and $\dfrac{f(z_0)}{h(z_0)} \neq 0$. So $g(z) = (z - z_0)^k \sum_{n=0}^{\infty} b_n (z - z_0)^n$ where $b_0 \neq 0$,

and it follows that g has a zero of order k at z_0.

(6.) Integrate $f(z) = \dfrac{z^2}{(z^2+1)(z^2+9)}$ over



The contour $\eta^R = \eta_1^R + \eta_2^R$ shown in the diagram. We have

$$\left| \int_{\eta_2^R} f(z)\, dz \right| \leq (\text{length } \eta_2^R) \cdot \sup_{|z|=R} f(z)$$

$$\leq \pi R \left(\frac{R^2}{(R^2-1)(R^2-9)} \right) = \frac{\pi R^3}{R^4 - 10R^2 + 9} \to 0 \text{ as } R \to \infty,$$

and

$$\int_{\eta_1^R} f(z)\, dz = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+9)}\, dx \to \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\, dx = 2 \int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\, dx$$

as $R \to \infty$.

Also $\int_{\eta^R} f(z)\,dz = 2\pi i\left[\mathrm{Res}_f(i) + \mathrm{Res}_f(3i)\right]$ by the residue theorem, since i and $3i$ are the only poles of f within η^R. Now $\mathrm{Res}_f[i] = \dfrac{\left(\frac{i^2}{i^2+9}\right)}{\frac{d}{dz}(z^2+1)|_i} = \dfrac{\left(\frac{-1}{8}\right)}{2i} = \dfrac{i}{16}$ and

$$\mathrm{Res}_f(3i) = \frac{\frac{(3i)^2}{(3i)^2+1}}{\frac{d}{dz}(z^2+9)|_{3i}} = \frac{\frac{-9}{-8}}{6i} = -\frac{3i}{16}, \text{ so}$$

$$\int_{\eta^R} f(z)\,dz = 2\pi i\left(\frac{i}{16} - \frac{3i}{16}\right) = \frac{\pi}{4} \quad \text{for all } R.$$

So $\frac{\pi}{4} = \int_{\eta_R} f = \lim_{R\to\infty} \int_{\eta_R} f = \lim_{R\to\infty}\left(\int_{\eta_1^R} f + \int_{\eta_2^R} f\right) = 2\int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)}\,dx$,

and $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{8}$.

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7 (a) If $z_0 \in S$ then $\lim_{z\to z_0} |f(z)| = \infty$, so $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.

So $\frac{1}{f}$ is bounded on a neighborhood of z_0, and therefore has a removable singularity at z_0. Moreover the holomorphic extension g of $\frac{1}{f}$ must have $g(z_0) = 0$.

(b) Since $f(\omega) = 0$ and the zeroes of f are isolated*, there exists $\varepsilon > 0$ s.t. $f(z) \neq 0$ for $z \in D(\omega, \varepsilon) \setminus \{\omega\}$

Then $\frac{1}{f}$ is holomorphic on $D(\omega,\varepsilon)\setminus\{\omega\}$, and $\lim_{z\to\omega}\left|\frac{1}{f(z)}\right| = \infty$ since $\lim_{z\to\omega} f(z) = f(\omega) = 0$. So ω is a pole of $\frac{1}{f}$.

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* otherwise f would be identically 0 on U, contradicting the fact that f has poles at the element(s) of S.

(c) Let $T = \{w \in U : f(w) = 0\}$. Then $g = \frac{1}{f}$ is defined and holomorphic on $U \setminus (T \cup S)$, and by part (a) g can be extended to a holomorphic function on $U \setminus T$ by defining $g(z) = 0$ for $z \in S$. Since g is holomorphic and not identically zero on the open, connected set $U \setminus T$, then the zeroes of g cannot accumulate, so S has no accumulation points in $U \setminus T$. Also, S cannot have any accumulation points in T, because if $w \in T$ then $|f(z)| \le 1$ in some neighborhood of w, and hence f cannot have any poles in that neighborhood. Therefore S cannot have any accumulation points in U.

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(8) We know from the fundamental theorem of algebra that a polynomial of degree n must have n zeroes in $\mathbb{C}$, counting multiplicity. Since the multiplicity of a zero of a polynomial is the same as the order of the zero, then the sum of the orders of the zeroes of p in $\mathbb{C}$ must be n. But all these zeroes lie within $D = \{|z| < R\}$, so by the argument principle

$$\int_{\partial D} \frac{p'(z)}{p(z)}\, dz = 2\pi i \cdot n$$
