Complex Analysis I
Exam 1

1. Prove that if z and w are complex numbers with $|z| = 1$, then

$$\left| \frac{z - w}{1 - \overline{z}w} \right| = 1.$$

2. Prove that if $f(z)$ is analytic, then $\overline{f(\overline{z})}$ is analytic.

3. Let $f(z) = |z|^2 = x^2 + y^2$.
a. Show that $f'(z)$ exists at $z = 0$.
b. Show that $f'(z)$ does not exist at any point z where $z \neq 0$.

4. Show that if f is holomorphic on a connected set U and $f(z)$ is real for all $z \in U$, then f is constant on U.

5. Let γ denote the circle $\{|z| = 1\}$, parameterized as a positively oriented simple closed curve.
a. Show that if $z = x + iy$ and $z \in \gamma$, then $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$.
b. Use the formula in part **a** to compute $\int_\gamma x \, dz$.

6. Let γ denote the circle $\{|z| = 2\}$, parameterized as a positively oriented simple closed curve.
a. Evaluate $\displaystyle\int_\gamma \frac{1}{z^2 - 1} \, dz$.
b. Suppose n is a positive integer, and evaluate $\displaystyle\int_\gamma \frac{e^z}{z^n} \, dz$. (You may assume that e^z is an entire function and $\frac{d}{dz}e^z = e^z$.)