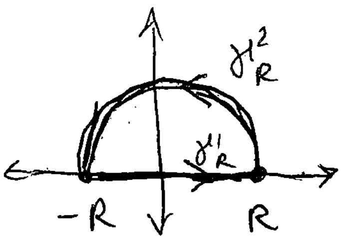
$\boxed{\text{ch. 4 \#47}}$ Let $g(z) = \dfrac{e^{iz}}{z^4+1}$ and let $\gamma_R = \gamma_R^1$ followed by γ_R^2

where $\gamma_R^1(t) = t$ $\;(-R \le t \le R)$ and $\gamma_R^2(t) = Re^{it}$ $\;(0 \le t \le \pi)$



For $z = x+iy \in \{\gamma_R^2\}$ we have, since $y > 0$,

$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}e^{-y}| = e^{-y} \le 1$ and

$|z^4+1| \ge |z|^4 - 1 = R^4 - 1$, so $|g(z)| \le \dfrac{1}{R^4-1}$.

Therefore $\left|\int_{\gamma_R^2} g(z)\,dz\right| \le (\text{length of } \gamma_R^2) \cdot \dfrac{1}{R^4-1} = \dfrac{\pi R}{R^4-1} \to 0$

as $R \to \infty$. Also $\displaystyle\int_{\gamma_R^1} g(z)\,dz = \int_{-R}^{R} \frac{\cos x + i\sin x}{x^4+1}\,dx = \int_{-R}^{R} \frac{\cos x}{x^4+1}\,dx$

(notice $\displaystyle\int_{-R}^{R} \frac{\sin x}{x^4+1}\,dx = 0$, as $\dfrac{\sin x}{x^4+1}$ is an odd function), so

$\displaystyle\lim_{R\to\infty} \int_{\gamma_R^1} g(z)\,dz = \int_{-\infty}^{\infty} \frac{\cos x}{x^4+1}\,dx$. Therefore

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1}\,dx = \lim_{R\to\infty}\left[\int_{\gamma_R^1} g(z)\,dz + \int_{\gamma_R^2} g(z)\,dz\right] = \lim_{R\to\infty} \int_{\gamma_R} g(z)\,dz.$$

On the other hand, by the residue Theorem, since for all $R > 1$

the ~~residues~~ poles of $g(z)$ within γ_R are at $\alpha = \dfrac{1+i}{\sqrt{2}}$ and $\beta = \dfrac{-1+i}{\sqrt{2}}$,

then $\displaystyle\int_{\gamma_R} g(z)\,dz = 2\pi i\left[\operatorname{Res}_g(\alpha) + \operatorname{Res}_g(\beta)\right]$ for all $R > 1$.

Now $\operatorname{Res}_g(\alpha) = \dfrac{e^{i\alpha}}{4\alpha^3} = \dfrac{e^{i\alpha}}{4\beta}$ and $\operatorname{Res}_g(\beta) = \dfrac{e^{i\beta}}{4\beta^3} = \dfrac{e^{i\beta}}{4\alpha}$, so $\operatorname{Res}_g(\alpha) + \operatorname{Res}_g(\beta) =$

$$= \frac{1}{4}\left[\frac{e^{i\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}}{(-1+i)/\sqrt{2}} + \frac{e^{i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}}{(1+i)/\sqrt{2}}\right] = \frac{\sqrt{2}}{4}\left[e^{-1/\sqrt{2}}\right]\left[e^{i/\sqrt{2}}\left(\frac{-1-i}{2}\right) + e^{-i/\sqrt{2}}\left(\frac{1-i}{2}\right)\right]$$

$$= \frac{1}{2\sqrt{2}}\, e^{-1/\sqrt{2}}\,[-i]\left[\cos\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right)\right]$$

Therefore, for all $R>1$,

$$\int_{\gamma_R} g(z)\,dz = (2\pi i)\left(\frac{1}{2\sqrt{2}}\right) e^{-1/\sqrt{2}}(-i)\left(\cos\left(\tfrac{1}{\sqrt{2}}\right)+\sin\left(\tfrac{1}{\sqrt{2}}\right)\right)$$

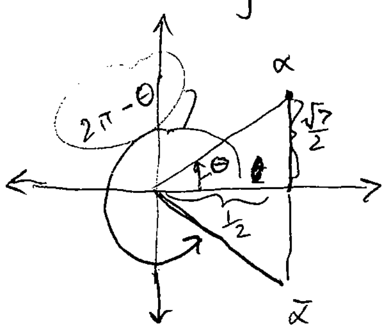
$$= \boxed{\frac{\pi}{\sqrt{2}}\, e^{-1/\sqrt{2}}\left(\cos\left(\tfrac{1}{\sqrt{2}}\right)+\sin\left(\tfrac{1}{\sqrt{2}}\right)\right)}$$

and hence this is the value of $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1}\,dx$.

$\boxed{\text{ch. 4 \#49}}$ Let $g(z) = \dfrac{\log(z)}{z^3+z+2}$, where $\log(z) = \log|z| + i\theta$ whenever $z = |z|e^{i\theta}$ for $0<\theta<2\pi$. The poles of g in $\mathbb{C}$ are at $z=-1$ and $z=\alpha$ and $z=\bar{\alpha}$ where $\alpha = \frac{1}{2}+i\frac{\sqrt{7}}{2}$

We have $\operatorname{Res}_g(-1) = \dfrac{\log(-1)}{3(-1)^2+1} = \dfrac{i\pi}{4}$, $\operatorname{Res}_g(\alpha) = \dfrac{\log\alpha}{3\alpha^2+1} = \dfrac{\log(\sqrt{2})+i\arctan\sqrt{7}}{\beta}$

and $\operatorname{Res}_g(\bar{\alpha}) = \dfrac{\log\bar{\alpha}}{3\bar{\alpha}^2+1} = \dfrac{\log(\sqrt{2})+i(2\pi-\arctan\sqrt{7})}{\bar{\beta}}$ where



$\theta = \arctan\sqrt{7} \in (0,\frac{\pi}{2})$ and $\beta = 3\alpha^2+1 = -\frac{7}{2}+i\frac{3\sqrt{7}}{2}$.

Hence the sum of the residues at these poles is

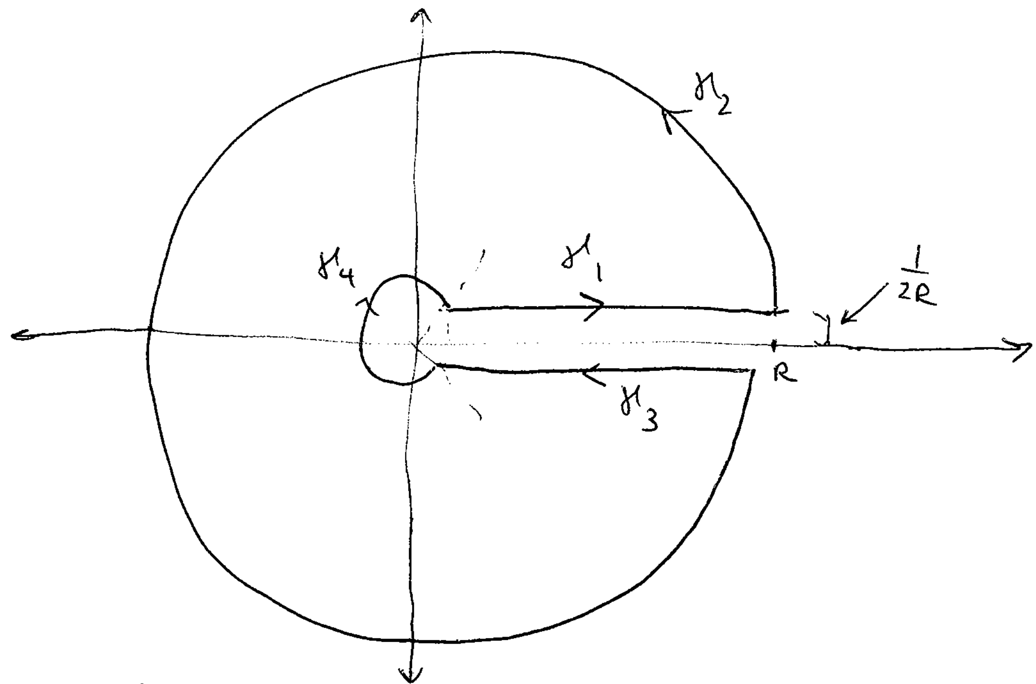
$$\frac{i\pi}{4} + \log(\sqrt{2})\left[\frac{1}{\beta}+\frac{1}{\bar{\beta}}\right] + \frac{2\pi i}{\bar{\beta}} + i(\arctan\sqrt{7})\left(\frac{1}{\beta}-\frac{1}{\bar{\beta}}\right).$$ Since (*)

$\dfrac{1}{\beta}+\dfrac{1}{\bar{\beta}} = \dfrac{\beta+\bar{\beta}}{|\beta|^2} = \dfrac{-7}{28} = \dfrac{-1}{4}$ and $\dfrac{1}{\beta}-\dfrac{1}{\bar{\beta}} = \dfrac{\bar{\beta}-\beta}{|\beta|^2} = \dfrac{-3\sqrt{7}\,i}{28}$, this

simplifies to $$-\frac{\log(\sqrt{2})}{4} + \frac{3\sqrt{7}\arctan\sqrt{7}}{28} - \frac{3\pi\sqrt{7}}{28}$$

(*) and $\dfrac{2\pi i}{\bar{\beta}} = \dfrac{-\pi i}{4} - \dfrac{3\pi\sqrt{7}}{28}$

Now let $\gamma_R = \gamma_R^1 + \gamma_R^2 + \gamma_R^3 + \gamma_R^4$ where γ_R^i are as shown:



$$\gamma_1(t) = t + i\frac{1}{2R} \qquad \left(\frac{1}{2R} \le t \le a, \text{ where } \sqrt{a^2+\left(\tfrac{1}{2R}\right)^2} = R\right)$$

$$-\gamma_3(t) = t - i\frac{1}{2R} \qquad \left(\frac{1}{2R} \le t \le a\right)$$

$$\gamma_2(t) = Re^{it} \qquad \left(\beta \le t \le 2\pi - \beta, \text{ where } \tan\beta = \frac{1/2R}{a}\right)$$

$$-\gamma_4(t) = \frac{1}{R}e^{it} \qquad \left(\frac{\pi}{4} \le t \le \frac{3\pi}{4}\right)$$

As in Example 4.6.5 in the text (see class notes for details) we have $\int_{\gamma_2} g(z)\,dz \to 0$ and $\int_{\gamma_4} g(z)\,dz \to 0$ as $R \to \infty$; and $\int_{\gamma_1} g(z)\,dz + \int_{\gamma_3} g(z)\,dz \longrightarrow (-2\pi i)\int_0^\infty \frac{dx}{x^3+x+2}$.

Also, for R large, all three poles of g are within γ_R, so

$$\lim_{R\to\infty} \int_{\gamma_R} g(z)\,dz = 2\pi i\left[-\frac{\log(\sqrt{2})}{4} + \frac{3\sqrt{7}\arctan\sqrt{7}}{28} - \frac{3\pi\sqrt{7}}{28}\right].$$

Since

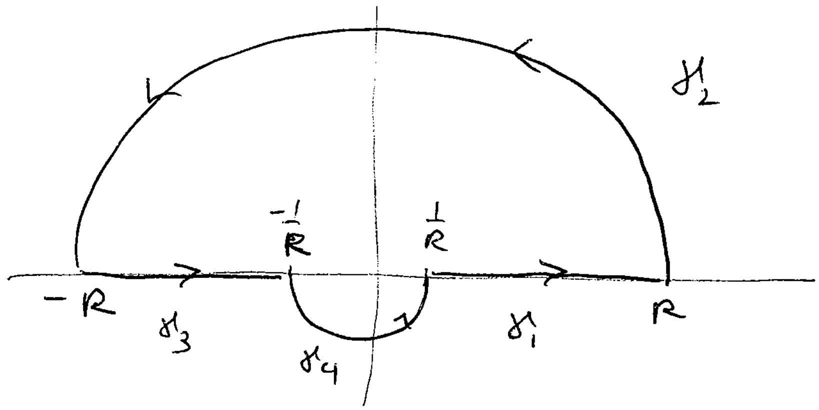
$$\lim_{R\to\infty}\int_{\gamma_R} g(z)\,dz = \lim_{R\to\infty} \int_{\gamma_1} g + \int_{\gamma_2} g + \int_{\gamma_3} g + \int_{\gamma_4} g = (-2\pi i)\int_0^\infty \frac{dx}{x^3+x+2},$$

it follows that

$$\int_0^\infty \frac{dx}{x^3+x+2} = \left[+\frac{\log(\sqrt{2})}{4} - \frac{3\sqrt{7}\arctan\sqrt{7}}{28} + \frac{3\pi\sqrt{7}}{28}\right]$$

$\boxed{\text{ch. 4 \#55}}$ Let $g(z) = \dfrac{1-e^{2iz}}{2z^2}$ and let $\gamma_R = \gamma_R^1 + \gamma_R^2 + \gamma_R^3 + \gamma_R^4$

where γ_R^i are as shown:



$\gamma_1(t) = t \quad (\frac{1}{R} \le t \le R)$; $\gamma_2(t) = Re^{it} \quad (0 \le t \le \pi)$,

$\gamma_3(t) = t \quad (-R \le t \le -\frac{1}{R})$; $\gamma_4(t) = \frac{1}{R}e^{it} \quad (\pi \le t \le 2\pi)$.

The only pole of g within γ_R is at $z=0$. There we have the Laurent expansion

$$g(z) = \frac{1}{2z^2}\left(1 - \left(1 + 2iz + \frac{(2iz)^2}{2!} + \frac{(2iz)^3}{3!} + \dots\right)\right)$$

$$= \frac{1}{2z^2}\left(-2iz - \frac{(2i)^2}{2}z^2 - \frac{(2i)^3}{6}z^3 - \dots\right)$$

$$= \frac{-i}{z} + 1 + \frac{2i}{3}z + \dots$$

So $\mathrm{Res}_g(0) = -i$, and by the Residue Theorem

$$\int_{\gamma_R} g(z)\,dz = 2\pi i(-i) = 2\pi, \text{ for all } R>0.$$

Now for $z \in \{\gamma_2(t)\}$ we have $z = x+iy$ with $y>0$, so

$$|g(z)| = \left|\frac{1-e^{2iz}}{2z^2}\right| \le \frac{1+|e^{2iz}|}{2R^2} = \frac{1+|e^{2i(x+iy)}|}{2R^2} = \frac{1+|e^{2ix}||e^{-2y}|}{2R^2}$$

$$\le \frac{1+1\cdot 1}{2R^2} = \frac{1}{R^2},$$

So $\left|\int_{\gamma_R^2} g(z)\,dz\right| \le \left(\text{length of } \gamma_R^2\right)\frac{1}{R^2} = \frac{\pi R}{R^2} = \frac{\pi}{R} \to 0$ as $R \to \infty$.

Also $\int_{\gamma_R^4} g(z)\,dz = \int_{\pi}^{2\pi} \frac{1-e^{2i(\frac{1}{R}e^{it})}}{2\left(\frac{1}{R}e^{it}\right)^2}\,\frac{1}{R}ie^{it}\,dt$, or

$$\int_{\gamma_R^4} g(z)\,dz = i\int_{\pi}^{2\pi}\left(\frac{1-e^{2i(\frac{1}{R}e^{it})}}{2\frac{1}{R}e^{it}}\right)dt.$$

Now we can use the fact (mentioned in class, or see a proof below) that there exists a constant $C>0$ such that $|1-e^z| \le C|z|$ for all $z \in D(0,1)$. It follows that for $R \ge 1$,

$\left|\frac{1-e^{2i(\frac{1}{R}e^{it})}}{2\frac{1}{R}e^{it}}\right| \le \frac{C\left|\frac{2}{R}e^{it}\right|}{\left|\frac{2}{R}e^{it}\right|} = C$, So by the "Bounded Convergence Theorem" (or the Lebesgue Dominated Convergence Theorem),

$$\lim_{R\to\infty}\int_{\gamma_R^4} g(z)\,dz = \lim_{R\to\infty} i\int_{\pi}^{2\pi}\left(\frac{1-e^{2i(\frac{1}{R}e^{it})}}{2\frac{1}{R}e^{it}}\right)dt$$

$$= i\int_{\pi}^{2\pi}\lim_{R\to\infty}\left(\frac{1-e^{2i(\frac{1}{R}e^{it})}}{2\frac{1}{R}e^{it}}\right)dt$$

$$= i\int_{\pi}^{2\pi}\lim_{w\to 0}\left(\frac{1-e^{2iw}}{2w}\right)dt$$

$$= i\int_{\pi}^{2\pi}\lim \frac{1}{2w}\left(1-\left(1+2iw+\frac{(2iw)^2}{2}+\dots\right)\right)dt = i\int_{\pi}^{2\pi}\lim_{w\to 0}\left(\frac{-2i}{2}+\frac{-(2i)^2}{4}w+\dots\right)dt$$

$$= i\int_{\pi}^{2\pi}\frac{-2i}{2}\,dt = (+1)\int_{\pi}^{2\pi}dt = (+\pi).$$

Finally, $\lim_{R\to\infty}\left[\int_{\gamma_3} g(z) + \int_{\gamma_1} g(z)\,dz\right]$

$$= \lim_{R\to\infty}\left[\int_{-R}^{-\frac{1}{R}} \frac{1-\cos 2x - i\sin 2x}{2x^2}\,dx + \int_{\frac{1}{R}}^{R} \frac{1-\cos 2x - i\sin 2x}{2x^2}\,dx\right]$$

$$= \lim_{R\to\infty}\left[2\int_{\frac{1}{R}}^{R} \frac{1-\cos 2x}{2x^2}\,dx\right]$$ (because $\frac{1-\cos 2x}{2x^2}$ is even and $\frac{\sin 2x}{2x^2}$ is odd)

$$= 2\int_0^\infty \frac{1-\cos 2x}{2x^2}\,dx = 2\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx.$$

Therefore

$$2\pi = \lim_{R\to\infty}\int_{\gamma_R} g(z)\,dz = \lim_{R\to\infty}\left\{\int_{\gamma^1} g(z)\,dz + \int_{\gamma^2} g\,dz + \int_{\gamma^3} g\,dz + \int_{\gamma_4} g\,dz\right\}$$

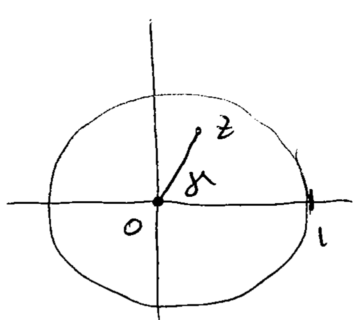
$$= \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx + 0 + \pi,$$

and we conclude that $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = 2\pi - \pi = \pi$.

Proof that $\exists C>0$ s.t. $\forall z \in D(0,1)$, $|1-e^z| \le C|z|$:

We claim we can take $C = e$. Let $z \in D(0,1)$ be given, and let γ be the straight-line path from the origin to z.

Then $|e^z - 1| = \left|\int_\gamma e^s\,ds\right|$ (since $\frac{d}{ds}(e^s) = e^s$)



$\le (\text{length of }\gamma)\left(\max_{|s|\le 1} |e^s|\right)$

$= |z|\cdot e$, [because for all $|s|\le 1$, $|e^s| = |e^{(x+iy)}| = e^x$ where $-1\le x\le 1$, so $|e^s| \le e^1 = e$.]

ch 5 #6

Suppose $f: D(0,1) \to \mathbb{C}$ is holomorphic and nonvanishing.

Then $h(z) = \frac{f'(z)}{f(z)}$ is holomorphic on $D(0,1)$, so by a previous result from class, h has a holomorphic antiderivative H on $D(0,1)$. Also, for any constant $C \in \mathbb{Z}$, $H(z)+C$ is also an antiderivative of h on $D(0,1)$.

Since $f(0) \neq 0$, then there exists $w \in \mathbb{Z}$ such that $e^w = f(0)$. Let $C = w - H(0)$; then $e^{H(0)+C} = e^w = f(0)$. Define $g: D(0,1) \to \mathbb{C}$ by $g(z) = H(z)+C$; Then g is an antiderivative of $h(z)$ on $D(0,1)$ and $e^{g(0)} = e^{H(0)+C} = f(0)$.

Now let $p(z) = \frac{e^{g(z)}}{f(z)}$. Then $p'(z) = \frac{f(z)\cdot e^{g(z)}\cdot g'(z) - e^{g(z)}\cdot f'(z)}{(f(z))^2} =$

$= \frac{e^{g(z)}}{(f(z))^2}\left[f(z)\cdot g'(z) - f'(z)\right] = 0$ since $f\cdot g' = f'$ on $D(0,1)$.

So p is constant on $D(0,1)$. Since $p(0) = \frac{e^{g(0)}}{f(0)} = 1$, then $p(z) = 1$ for all $z \in D(0,1)$, so $e^{g(z)} = f(z)$ for all $z \in D(0,1)$.

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ch.5 #15 Here is a useful Lemma for taking complex derivatives inside complex line integrals:

Lemma: Suppose $U \subseteq \mathbb{C}$ is open, and $[a,b] \subseteq \mathbb{R}$. Suppose $\varphi(z,t)$ is a complex-valued function defined for $z \in U$ and $t \in [a,b]$ such that

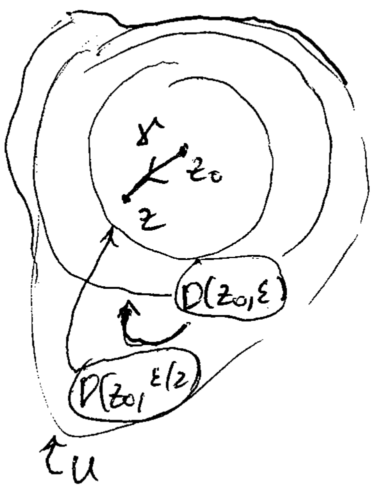
(i) For each $z_0 \in U$, $\varphi(z_0,t)$ is a continuous function from $[a,b]$ to $\mathbb{C}$, and

(ii) For each $t_0 \in [a,b]$, $\varphi(z,t_0)$ is a holomorphic function on U.

Define $f(z) = \int_a^b \varphi(z,t)\,dt$ for $z \in U$.

Then f is holomorphic on U, and $f'(z) = \int_a^b \frac{d}{dz}\varphi(z,t)\,dt$.

Proof of Lemma: Suppose $z_0 \in \mathcal{U}$. Then there exists $\varepsilon > 0$ such that $\overline{D(z_0,\varepsilon)} \subseteq \mathcal{U}$. For each $z \in D(z_0,\frac{\varepsilon}{2})$, let γ be the straight-line path from z_0 to z.



Then $\forall t \in [a,b]$
$$\left| \frac{\varphi(z,t)-\varphi(z_0,t)}{z-z_0} \right| = \left| \frac{\int_\gamma \varphi'(\zeta,t)\,d\zeta}{(z-z_0)} \right|$$

$$\leq \frac{(\text{length of } \gamma)}{|z-z_0|} \sup_{\substack{\zeta \in \overline{D(z_0,\varepsilon/2)} \\ t\in[a,b]}} |\varphi'(\zeta,t)| = \sup_{\substack{\zeta \in \overline{D(z_0,\varepsilon/2)} \\ t\in[a,b]}} |\varphi'(\zeta,t)|$$

where $\varphi'(\zeta,t)$ denotes the derivative of $\varphi(z,t)$ with respect to z, evaluated at ζ.

From the Cauchy estimates we see that for all $\zeta \in \overline{D(z_0,\frac{\varepsilon}{2})}$,

$$|\varphi'(\zeta,t)| \leq \frac{M}{\varepsilon/2}, \text{ where } M = \sup_{\substack{\zeta \in \overline{D(z_0,\varepsilon)} \\ t\in[a,b]}} |\varphi(\zeta,t)|.$$

Since φ is continuous on the compact set $\overline{D(z_0,\varepsilon)} \times [a,b]$, then $M < \infty$. So we have shown that for all $z \in \overline{D(z_0,\varepsilon/2)}$ and all $t \in [a,b]$, $\left| \frac{\varphi(z,t)-\varphi(z_0,t)}{z-z_0} \right| \leq \frac{M}{(\varepsilon/2)}$.

It now follows from the Bounded (or Dominated) Convergence Theorem that
$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{\int_a^b \varphi(z,t)\,dt - \int_a^b \varphi(z_0,t)\,dt}{z-z_0} =$$

$$= \lim_{z\to z_0} \int_a^b \left[\frac{\varphi(z,t)-\varphi(z_0,t)}{z-z_0} \right] dt = \int_a^b \lim_{z\to z_0} \left[\frac{\varphi(z,t)-\varphi(z_0,t)}{z-z_0} \right] dt = \int_a^b \frac{\partial \varphi}{\partial z}(z_0,t)\,dt$$

So $f'(z_0)$ exists and equals $\int_a^b \frac{\partial\varphi}{\partial z}(z_0,t)\,dt$ for all $z_0 \in \mathcal{U}$. It now follows from Goursat's Theorem (Appendix B) that since f is differentiable on $\mathcal{U}$ then f is holomorphic on $\mathcal{U}$. //

(a) Now we can apply the Lemma to $N(z)$. Suppose $\mathcal{U} \subseteq \mathbb{C}$ is open, $\overline{D(P,r)} \subseteq \mathcal{U}$, and $Q = f(P)$, and $f(z) \neq Q$ for $z \in \overline{D(P,r)} \setminus \{P\}$.

Since f is continuous on $\mathcal{U}$ and $\partial D(P,r) \subseteq \mathcal{U}$ is compact, then $f(\partial D(P,r))$ is compact. Since $Q \notin f(\partial D(P,r))$, there exists $\varepsilon > 0$ s.t. $\overline{D(Q,\varepsilon)}$ does not intersect $f(\partial D(P,r))$.

Thus, for $z \in D(Q,\varepsilon)$, $f(\zeta) - z \neq 0$ for all $\zeta \in \partial D(P,r)$.

Now let $\varphi(z,t) = \left(\frac{1}{2\pi i}\right) \dfrac{f'(\zeta(t))}{f(\zeta(t)) - z}$, where $\zeta(t)$ is a C^1 parametrization of $\partial D(P,r)$ (for example, $\zeta(t) = P + re^{it}$ for $0 \le t \le 2\pi$.)

Then the hypotheses (i) and (ii) of the Lemma are satisfied by $\varphi(z,t)$ on $\tilde{\mathcal{U}} = D(Q,\varepsilon)$ and $[a,b] = [0,2\pi]$; so we obtain that the function

$$N(z) = \frac{1}{2\pi i} \int_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta) - z}\, d\zeta = \int_0^{2\pi} \varphi(z,t)\, dt$$

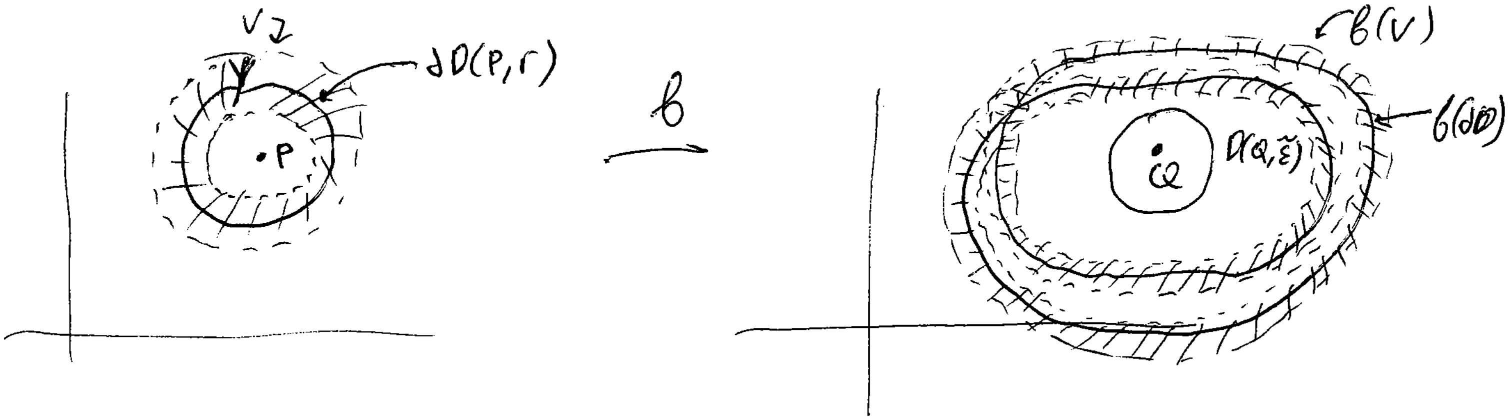
is holomorphic on $D(Q,\varepsilon)$ and has derivative

$$N'(z) = \int_0^{2\pi} \frac{d}{dz} \varphi(z,t)\, dt = \frac{1}{2\pi i} \int_{\partial D(P,r)} \frac{d}{dz}\left(\frac{f'(\zeta)}{f(\zeta) - z}\right) d\zeta$$

$$= \frac{1}{2\pi i} \int_{\partial D(P,r)} \frac{f'(\zeta)}{(f(\zeta) - z)^2}\, d\zeta$$

(b) As Elizabeth pointed out, the problem is mis-stated: it should be: "There exists $\tilde{\varepsilon} > 0$ and a neighborhood V of $\partial D(P,r)$ such that for each fixed $z \in D(Q,\tilde{\varepsilon})$, there exists a holomorphic function $H(\zeta)$ on V such that

$$H'(\zeta) = \frac{f'(\zeta)}{(f(\zeta)-z)^2} \quad \text{for } \zeta \in V."$$



To see this, just choose V open s.t. $\partial D(P,r) \subseteq \overline{V} \subseteq U \setminus \{P\}$ and $Q \notin f(\overline{V})$. Then $f(\overline{V})$ is compact and so there exists $\tilde{\varepsilon} > 0$ such that $D(Q,\tilde{\varepsilon})$ does not intersect $f(\overline{V})$.

Then define $H(\zeta) = \dfrac{-1}{(f(\zeta)-z)}$,

Since $f(\zeta) \neq z$ for $\zeta \in V$ and $z \in D(Q,\tilde{\varepsilon})$, then this $H(\zeta)$ has the desired property.

c Since $H(\zeta)$ is holomorphic on a neighborhood V of $\partial D(P,r)$, it follows from the Fundamental Theorem of line integrals (Prop. 2.1.6 in the text) that

$$\frac{1}{2\pi i}\int_{\partial D(P,r)} \frac{f'(\zeta)}{(f(\zeta)-z)^2}\,d\zeta = \frac{1}{2\pi i}\int_{\partial D(P,r)} H'(\zeta)\,d\zeta = 0,$$

and hence from part a that $N'(z)=0$ for all $z \in D(Q,\tilde{\varepsilon})$ ← [This was another misstatement in the text].

d Since $N'(z)=0$ on $D(Q,\tilde{\varepsilon})$ then $N(z)$ is constant on $D(Q,\tilde{\varepsilon})$.

$\boxed{\text{ch. 5 \#16}}$

Since $f'(P)=0$, then the order k of f at P is greater than or equal to 2. Let V be any neighborhood of P. Then by the local mapping principle [applied to V instead of $\mathcal{U}$] (cf. Theorem 5.2.2, or the versions given in class), there exist $\delta>0$ and $\varepsilon>0$ such that for every $w \in D(f(P),\varepsilon)\setminus\{f(P)\}$, there are at least two values of z in $D(P,\delta)\subseteq V$ such that $f(z)=w$. This means that f is not one-to-one in $D(P,\delta)$ and hence f is not one-to-one in V.