Solutions to problems on Assignment 7

9. We will prove the converse of the desired statement. That is, we assume there exists $N \in \mathbf{Z}$ such that for every sequence z_n in $D(P,r)\backslash\{P\}$ with $\lim z_n = P$, there exists $n \in \mathbf{N}$ such that $|(z_n - P)^N f(z_n)| \leq N$; and we will show that f cannot have an essential singularity at P.

From our assumption it follows that there exists some $r_0 \in (0,r)$ such that for every $z \in D(P,r_0)\backslash\{P\}$, $|(z-P)^N f(z)| \leq N$. For if this were not true, then for every $n \in \mathbf{N}$, there would exist $z = z_n \in D(P,1/n)\backslash\{P\}$ such that $|(z_n - P)^N f(z_n)| > N$. Then $\{z_n\}$ is a sequence such that $\lim z_n = P$ and there is no $n \in \mathbf{N}$ such that $|(z_n - P)^N f(z_n)| \leq N$, violating our assumption.

Now define $g(z) = (z-P)^N f(z)$. From the preceding paragraph we know that $g(z)$ is bounded on $D(P,r_0)\backslash\{P\}$, so g has a removable singularity at P. Therefore $g(z)$ has a power series expansion

$$g(z) = \sum_{n=0}^{\infty} a_k (z-P)^k$$

on $D(P,r_0)\backslash\{P\}$. Hence

$$f(z) = g(z)/(z-P)^N = \sum_{n=0}^{\infty} a_k (z-P)^{k-N}$$

on $D(P,r_0)\backslash\{P\}$. But this implies that f has a pole of order N at P, not an essential singularity at P.

23. Since f has a pole of order k at P, then f has the Laurent expansion

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-P)^n$$

for all z in some punctured neighborhood $D(P,r)\backslash\{P\}$. Then $g(z) = (z-P)^k f(z)$ has the expansion

$$g(z) = \sum_{n=-k}^{\infty} a_n (z-P)^{n+k} = \sum_{n=0}^{\infty} a_{n-k} (z-P)^n$$

in $D(P,r)\backslash\{P\}$. So the coefficient of $(z-P)^n$ in the Taylor series expansion for g is the same as the coefficient of $(z-P)^{n-k}$ in the Laurent series expansion for f.

34(a). We are integrating f over the circle $C = \{|z| = 5\}$ with (presumably) the positive orientation. The poles of f are at $z = -1$ and $z = -2i$, both of which are within C. The residue of f at $z = -1$ is $\dfrac{-1}{-1+2i} = \dfrac{1+2i}{5}$, and the residue of f at $z = -2i$ is $\dfrac{-2i}{-2i+1} = \dfrac{4-2i}{5}$. So by the residue theorem,

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1+2i}{5} + \frac{4-2i}{5} = 1.$$

34(d). The poles of f are at 0, -1, and -2, all of which are within γ. We have

$$\begin{aligned} \mathrm{Res}_f(0) &= e^0/((1)(2)) = 1/2 \\ \mathrm{Res}_f(-1) &= e^{-1}/((-1)(1)) = -1/e \\ \mathrm{Res}_f(-2) &= e^{-2}/((-2)(-1)) = 1/(2e^2) \end{aligned}$$

Since γ has the negative orientation, then the desired integral is equal to the negative of the sum of the residues, and is therefore equal to $-(e^2 - 2e + 1)/(2e^2)$.

34(i). We have $f(z) = \dfrac{\sin z}{\cos z}$, and $\sin z$ and $\cos z$ are entire, so the only singularities of f are at the zeroes of $\cos z$, which as we saw in class are all on the real line and are the same as the zeroes of the real cosine function, namely $\{((2k+1)\pi)/2 : k \in \mathbf{Z}\}$. All these poles are simple (because $\sin z$ is non-zero at each pole and the derivative of $\cos z$ is non-zero at each pole) and the residues are

$$\mathrm{Res}_f\left(\frac{(2k+1)\pi}{2}\right) = \frac{\sin((2k+1)\pi/2)}{\sin((2k+1)\pi/2)} = 1.$$

From the diagram of γ we see that the only poles of f about which γ has non-zero index are $-3\pi/2$ and $3\pi/2$, and $\mathrm{Ind}_\gamma(-3\pi/2) = -1$ and $\mathrm{Ind}_\gamma(3\pi/2) = 1$. Therefore

$$\frac{1}{2\pi i}\int_\gamma \tan z\; dz = \mathrm{Res}_f\left(\frac{-3\pi}{2}\right)\mathrm{Ind}_\gamma(-3\pi/2) + \mathrm{Res}_f\left(\frac{3\pi}{2}\right)\mathrm{Ind}_\gamma(3\pi/2) = -1 + 1 = 0.$$