THE REFINED RELATIVE COMPRESSION BODY NEIGHBORHOOD

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ABSTRACT. We construct the refined relative compression body neighborhood of a free side of a pared 3-manifold, and detail its properties. It is one of the topological ingredients needed for a theory of splittings of Kleinian groups currently under development. This theory extends previous work of Abikoff-Maskit and Maskit.

INTRODUCTION

This paper concerns the topological structure of pared 3-manifolds, which arise naturally in the study of hyperbolic 3-manifolds. A pared structure consists of disjoint incompressible annuli and tori in the boundary of the 3-manifold, which satisfy some strong additional conditions. Thurston's Geometrization Theorem asserts that these conditions are precisely what is needed to guarantee the existence of a hyperbolic structure on the interior of the manifold, for which the elements of the boundary pattern correspond to "cusps" of the hyperbolic structure. Convenient references for the topology of pared 3-manifolds are [3, 9].

Bonahon [2] originated the concept of a compression body neighborhood of a compressible boundary component of a 3-manifold. In [3], a relative version of this theory was developed, which associates "relative compression body" neighborhoods to the compressible free sides of a 3-manifold with boundary pattern $(M, \underline{\underline{m}})$.

The purpose of this paper is to construct a special type of relative compression body neighborhood for a free side F of a pared 3-manifold (M, P). It is called the refined relative compression body neighborhood of F. One may describe it as the minimal compression body neighborhood V so that any loop in F which is homotopic into the pared locus of M is homotopic in V into the pared locus of V.

In contrast to the relative compression body neighborhoods used in [3], it is not possible to choose disjoint refined relative compression body neighborhoods of the free sides of a pared manifold. We will see in theorem 6.2

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below, however, that there is a disjoint collection of refined relative compression body neighborhoods of *some* of the free sides whose union contains *all* free sides. The union is unique up to isotopy, and the closure of its complement is called the *refined core.* The refined core is a pared manifold which has the property that any essential loop in a free side which is homotopic into the pared locus is homotopic in that free side into the pared locus.

We will review the basic definitions and facts about relative compression bodies in section 1. Since we will be dealing only with the restricted case of pared manifolds, we will use a simplified version of the characteristic submanifold theory used in [3]. Section 2 contains background material on pared manifolds. Since the construction of the refined relative compression body neighborhood is rather complicated, we first give an outline of the process. This constitutes section 3, and the detailed version of the construction is given in section 4. The key "enclosing" properties of the refined relative compression body neighborhood are proven in section 5, and the main existence and uniqueness theorems constitute section 6.

At the current juncture, this paper is not intended for publication, but only as reference material that can be made publicly available. We anticipate that it will be incorporated into later work which will provide a splitting theory for Kleinian groups based on that developed by W. Abikoff and B. Maskit [1] and extended by Maskit [7, 8].

1. Relative compression bodies

For our current purposes, a relative compression body will mean a pair (V, S) which can be constructed as follows. For $1 \leq i \leq m$ let F_i be a connected orientable 2-manifold, not a 2-sphere or 2-disc. Form a connected irreducible 3-manifold V from $\bigcup_{i=1}^{m} F_i \times \mathrm{I}$ by attaching 1-handles to the manifold interior of $\bigcup_{i=1}^{m} F_i \times \{1\}$. Denote by F the union of the intersection of ∂V with $\bigcup_{i=1}^{m} F_i \times \{1\}$ and the intersection of ∂V with the 1-handles. We call (V, S) a *relative compression body* if either V is a handlebody and S is empty, or V is constructed as above and S is the union of $\bigcup_{i=1}^{m} \partial F_i \times \{1\}$ with a possibly empty subcollection of the surfaces $F_i \times \{0\}$.

Components of the closure of $\partial V - S$ are called *free sides.* In particular, F is called the *distinguished free side* of (V, S). We denote each $F_i \times \{0\}$ by F_i and call it a *constituent* of V.

More generally, if M is a compact orientable irreducible 3-manifold and P is an incompressible 2-manifold in ∂M, we call (M, P) a *manifold pair* and refer to the components of the closure $\overline{\partial M - P}$ as the *free sides* of (M, P). A map of a 2-manifold G into M is called *strongly admissible* if ∂G maps to the manifold interior of P, and homotopies of this map are called admissible if they preserve this condition. Unless otherwise stated, it is assumed that maps of 2-manifolds into M are proper, meaning that the preimage of ∂M is ∂G.

The next fact, adapted from lemma 3.1.1 of [3], shows that relative compression bodies contain only the obvious incompressible surfaces.

Lemma 1.1. *Let (V,S) be a relative compression body with distinguished free side F, and let G be a strongly admissible connected 2-manifold imbedded in (V,S), with $G \neq S^2$. Assume that $\pi_1(G) \to \pi_1(V)$ is injective. Then there is a unique constituent F_i of (V,S) such that G is admissibly isotopic to $F_i \times \{1/2\}$.*

Let F be a free side of a manifold pair (M,P). Suppose that (V,S) is a codimension-zero submanifold of (M,P) with $F \subset V$. We say that (V,S) is a *relative compression body neighborhood of F* if

(a) $S \subset P$ and $S \cap \partial P = \partial F$.
(b) (V,S) is a relative compression body with distinguished free side F,
(c) the frontier of V is incompressible in M, and

A relative compression body neighborhood (V,S) of F for which each constituent is properly imbedded is said to be *minimally imbedded.* From proposition 3.2.3 of [3], we have a strong existence and uniqueness property for such neighborhoods.

Proposition 1.2. *Let (M,P) be a manifold pair, and let $S_1,\ldots,\ S_r$ be a collection of free sides of (M,P). Then there exist disjoint minimally imbedded relative compression neighborhoods for the S_i. Their union is unique up to admissible ambient isotopy in (M,P).*

Here, an admissible isotopy of M means one for which the image of P is P at all times.

The minimal compression body neighborhood of F has the following enclosing property, lemma 3.2.2 of [3].

Lemma 1.3. *Let F be a free side of (M,P) and let (V,S) be a minimally imbedded relative compression body neighborhood of F in (M,P). Let V' be any irreducible codimension-zero submanifold of M which is a neighborhood of F having incompressible frontier. Then there is an admissible ambient isotopy of (M,P) that moves V into the topological interior of V'.*

Suppose that (V,S) is a relative compression body neighborhood of a free side F of (M,P). A component R of $\overline{M-V}$ is called *spurious* if $(R, R\cap\partial M)$ is of the form $(G\times[-1,0], \{G\times\{-1\}\cup\partial G\times[-1,0]\})$ where $G\times\{0\}$ is a constituent of V. Note that $R\cap\partial M$ is connected and homeomorphic to G. The union of V with R is still a relative compression body; the constituent G is replaced by a new constituent $\overline{\partial R - G}$. The *normally imbedded relative compression body neighborhood* of a free side F is the union of a minimally imbedded relative compression body neighborhood V of F with all spurious components of $\overline{M-V}$. From proposition 3.4.2 of [3], we have existence and uniqueness of such neighborhoods.

Proposition 1.4. *Let (M,P) be a manifold pair. Then there exist disjoint normally imbedded relative compression neighborhoods for the free sides of (M,P). Their union is unique up to admissible ambient isotopy of (M,P).*

Let $V(M)$ be the union of a collection of disjoint normally imbedded relative compression neighborhoods for the free sides of (M, P). Denote the closure of $M - V(M)$ by M', and put $P' = M' \cap P$. We call the manifold pair (M', P') the *normal core* of (M, P). By proposition 1.4, the normal core is unique up to ambient isotopy.

2. Pared 3-manifolds

Let (M, P) be a manifold pair with M not a 3-ball. We say that (M, P) is a *pared* 3-manifold (see Morgan [9]) if the following three conditions hold.

(P1) Every component of P is an incompressible torus or annulus.

(P2) Every noncyclic abelian subgroup of $\pi_1(M)$ is conjugate into the fundamental group of a component of P.

(P3) Every map $\phi\colon (\mathrm{S}^1 \times \mathrm{I}, \mathrm{S}^1 \times \partial\mathrm{I}) \to (M, P)$ which induces an injection on fundamental groups is homotopic, as a map of pairs, to a map ψ such that $\psi(\mathrm{S}^1 \times \mathrm{I}) \subset P$.

Let T^2 and A^2 denote the torus and annulus respectively. If $(M, P) = (T^2 \times \mathrm{I}, T^2 \times \{0\})$, or $(A^2 \times \mathrm{I}, A^2 \times \{0\})$, or $(A^2 \times \mathrm{I}, \underline{\underline{\emptyset}})$, then (M, P) is said to be *elementary,* otherwise it is *nonelementary.* The elementary pared 3-manifolds correspond to the hyperbolic 3-manifolds with abelian fundamental groups.

From [3] we have some properties of nonelementary pared 3-manifolds.

Lemma 2.1. *Let (M, P) be a nonelementary pared 3-manifold.*

(i) *Every toroidal component of ∂M is contained in P.*

(ii) *M is not homeomorphic to $T^2 \times I$, to the I-bundle over the Klein bottle, or to the solid torus.*

(iii) *M does not contain an incompressible Klein bottle.*

(iv) *For each component P_0 of P, the subgroup $\pi_1(P_0)$ is a maximal abelian subgroup of $\pi_1(M)$.*

Two pared 3-manifolds (M, P) and (N, Q) are called *pared homotopy equivalent* when they are homotopy equivalent as pairs, and a homotopy equivalence of pairs is called a *pared homotopy equivalence.*

By lemma 5.2.1 of [3], a pared manifold satisfies Johannson's conditions to have a characteristic submanifold if all of its free sides are incompressible. We will briefly review the characteristic submanifold here, as adapted to pared 3-manifolds. (In Johannson's language of boundary patterns, we will be describing the case of 3-manifolds with useful boundary patterns that are the completions of boundary patterns with disjoint elements).

Let (M, P) be a pared manifold. Denote $\overline{\partial M - P}$ by $-P$. A map of a surface G into M is called *admissible* if it is proper and does not meet ∂P. A map of a torus into M is called *essential* if it induces an injection on fundamental groups. A map of an annulus X into M is called *essential* if it is admissible, induces an injection on fundamental groups, and is not properly homotopic to a map into ∂M whose image meets at most one component of ∂P.

Let R be an I-bundle over a surface B. A component of the associated ∂I-bundle of R is called a *lid*. The closures in ∂R of the components of the complement of the lid or lids are called the *sides*; these are the restrictions of the I-bundle to the boundary components of B. An imbedding of R into M is called *admissible* if each lid lies either in P or in $-P$, and each side is either a component of P, or a component of $-P$, or is properly imbedded.

An embedded Seifert fibered space R in M is *admissibly embedded* if $R \cap \partial M$ is a 2-manifold which is a union of fibers in ∂R, and if whenever R meets ∂P, each component of $R \cap \partial P$ is a fiber of R and R contains a regular neighborhood in ∂M of this fiber.

An admissibly embedded I-bundle or Seifert fibered space R in M is *essential* if every component of the frontier of R in M is an essential torus or annulus in M. In particular, this implies that $\pi_1(R) \to \pi_1(M)$ is injective. A homotopy $F\colon R \times I \to M$ is *admissible* if for each t, $F_t^{-1}(P) = F_0^{-1}(P)$ and $F_t^{-1}(-P) = F_0^{-1}(-P)$.

A compact codimension-zero submanifold Σ of M has the *engulfing property* if every essential embedding $f\colon R \to M$ of an I-bundle or a Seifert fibered space into M is admissibly isotopic into Σ. We define Σ to be a *characteristic submanifold* of M if Σ consists of a collection of essential I-bundles and Seifert fiber spaces having the engulfing property, and Σ is minimal in the sense that no proper subcollection of the components of Σ has the engulfing property.

The characteristic submanifold has another useful property called the *enclosing property.* This means that every essential map of a torus, or annulus into M is admissibly homotopic to a map with image in Σ.

Jaco and Shalen [5] and Johannson [6] gave conditions sufficient to guarantee that a manifold pair (M, P) have a characteristic submanifold, in which case it is unique up to admissible isotopy. In [3], it is proven that these conditions are satisfied by pared manifolds provided that each free side is incompressible. Moreover, one may choose a fibering of the characteristic submanifold of (M, P) so that none of its components is an I-bundle over an annulus or Möbius band. The characteristic submanifold of a boundary pattern associated to a pared 3-manifold is then described by the following theorem from [3].

Theorem 2.2. (Pared Characteristic Submanifold Restrictions) *Let (M, P) be a nonelementary pared 3-manifold whose free sides are incompressible. Let Σ denote the characteristic submanifold of (M, P), with fibering selected so that no component of Σ is an I-bundle over an annulus or Möbius band.*

(i) *Suppose V is an I-bundle component of Σ. Then each of its lids lies in a free side of (M, P), its sides are components of P, and its base surface has negative Euler characteristic.*

(ii) *Suppose V is a Seifert fibered component of Σ. Then V is homeomorphic either to $T^2 \times I$ or to a solid torus. If V is homeomorphic to $T^2 \times I$, then one of its boundary components is a component of P and the other components of $V \cap \partial M$ are annuli in free sides of (M, P).*

In the pared setting, the I-bundles of the characteristic submanifold can appear only in very specific conigurations, described in lemma 5.3.1 from [3]:

Lemma 2.3. *Let (M, P) be a pared nonelementary 3-manifold with incompressible free faces. Let Σ be the characteristic submanifold of (M, P), and let V be a component of Σ which is an I-bundle. Then*

(i) *the lids of V must be contained in free sides of (M, P), and*
(ii) *every side of V which meets ∂M is a component of P.*

We will use this to develop the following special property of the characteristic submanifold of a pared manifold.

Lemma 2.4. *Let (M, P) be a nonelementary pared manifold with incompressible free faces, and let Σ be the characteristic submanifold of (M, P). Then every essential annulus in (M, P) with one end in a component of P is admissibly homotopic into a Seifert-fibered component of Σ. Every collection of disjoint imbedded essential annuli, each having one end in P, is admissibly isotopic into the union of the Seifert-fibered components of Σ.*

Proof. Let A be a singular annulus as in the lemma. Since A is essential, pared condition (P3) shows that its other end lies in $-P$. By the enclosing property of the characteristic submanifold, there is an admissible homotopy of A which moves it into a component V of Σ. Suppose for contradiction that V is I-fibered. Let G be the component of P that contains one end of A. Lemma 2.3 implies that G is a side of V. Since A is essential, the other end of A must also be in a side of V (rather than a lid), hence must lie in P, contradicting pared condition (P3) for (M, P). For a collection of disjoint imbedded essential annuli, the argument is the same except that the Engulfing Property is used in place of the Enclosing Property. $\square$

3. The basic idea of the construction

Roughly speaking, the refined relative compression body neighborhood is constructed by expanding the normally imbedded compression body V of F so that it contains a representative of each isotopy class of simple loop in P that lies in $\overline{M - V}$ and is homotopic into V. Unlike its minimally and normally imbedded cousins, the components of its frontier need not be constituents. The construction actually proceeds in two stages.

First, a certain union X of relative compression body neighborhoods of the free sides is constructed. It is called the pre-refined compression submanifold of (M, P). To obtain X, we start with a union $\bigcup V_i$ of disjoint, normally imbedded compression body neighborhoods for the free sides of (M, P). The closure of $M - \bigcup V_i$ is the normal core (M', P'). To obtain X, we add to $\bigcup V_i$

regular neighborhoods of a disjoint collection of annuli in M', each having one end in the frontier of $\bigcup V_i$ and the other end in P. These annuli lie in the Seifert-fibered components of the characteristic submanifold of (M', P').

Once X has been constructed, we can proceed with the second stage of the construction of the refined relative compression body neighborhood (W, Q) of a free side F of (M, P). Let W_0 be the component of X which contains F. Obtain W by adding to W_0 all complementary components that have the form of products $S \times \mathrm{I}$ whose intersection with W_0 consists exactly of $S \times \{0\}$. Such complementary components are said to be adherent. Note that these are components of the complement of W_0, not the complement of X. Indeed, some of the adherent components for W_0 may contain other components of X and hence other free sides, although we will see that these additional free sides must be incompressible. In fact, such a free side must have the form $S \times \{1\}$ for some product strcture $S \times \mathrm{I}$ on the adherent component, for which the component of X that contains $S \times \{1\}$ is simply $S \times [1/2, 1]$. Finally, we put $Q = W \cap P$.

The pre-refined compression submanifold X has two enclosing properties, developed in propositions 5.1 and 5.2. The first says that any annulus in (M, P) with one end in a free side and the other an essential loop in P is admissibly homotopic into X. The second says that if S is a component of the frontier of X, and γ is an essential closed curve in S which is homotopic in M into P, then γ is homotopic in S into $S \cap P$. The latter is not stated in the usual form of an enclosing property, but in the presence of the pared property it is equivalent to the assertion that any annulus with one end an essential loop γ in the frontier of X and the other end in P, is homotopic, relative to γ and keeping the other end in P, into the frontier. This equivalence is explained after the statement of proposition 5.2.

The enclosing properties of X lead immediately to corresponding enclosing properties of (W, Q). After proving proposition 5.3, which analyzes the adherent components, and noting that (W, Q) inherits the enclosing properties of X, we are prepared for the main results. The first is the uniqueness of (W, Q), theorem 6.1. It says that the second enclosing property, together with a couple of topological conditions, characterizes the refined relative compression body neighborhood up to admissible isotopy in (M, P). The second main result, theorem 6.2, shows that there is a unique way to select a disjoint collection of refined relative compression body neighborhoods of *some* of the free sides of (M, P), so that the union contains all the free sides. This union satisfies appropriate versions of the two enclosing properties.

4. The construction

In this section we will give the details of the construction of X. As in the section 2, for the manifold pair (M, P) we denote $\overline{\partial M - P}$ by $-P$.

Fix a union $\bigcup V_i$ of disjoint normally imbedded relative compressible body neighborhoods of the free sides of (M, P), and let (M', P') be the associated normal core. Then (M', P') is pared, and each free side of (M', P') is incompressible, so the characteristic submanifold Σ of (M', P') exists.

Consider a Seifert-fibered component Z of Σ. By the Pared Characteristic Submanifold Restrictions 2.2, Z is homeomorphic to either a solid torus or $T^2 \times \mathrm{I}$.

Suppose first that Z is a solid torus. If Z is disjoint from P', we ignore it. If not, then by pared condition (P3), Z meets P' in exactly one annulus A' of P' (and since Z is admissibly imbedded, one of the components of $Z \cap \partial M'$ will be a regular neighborhood of A'). A core circle of A' must generate $\pi_1(Z)$, for if not then $\overline{\partial Z - A'}$ gives a violation of (P3). Let $B_1, \ldots,$ B_k be the other annuli of $Z \cap \partial M'$, which must lie in $\overline{\partial M' - P'}$, the frontier of $\bigcup V_i$. For each B_j, choose an imbedded annulus C_j in Z having one boundary component a core circle of B_j and the other in the interior of A'. Choose the C_j to be disjoint; then, $\bigcup C_j$ is unique up to admissible isotopy in Z.

If Z is $T^2 \times \mathrm{I}$, then by the Pared Characteristic Submanifold Restrictions 2.2, one of its boundary components is a torus component of P' and the rest of $Z \cap \partial M'$ consists of annuli $B_1, \ldots, B_k$ in $\overline{\partial M' - P'}$. As in the case when Z was a solid torus, choose a disjoint collection of imbedded annuli C_j in Z each having one boundary component a core circle of one of the B_j and the other in the torus component of P'. Again, $\bigcup C_j$ is unique up to admissible isotopy in Z.

Notice that in both these cases, any annulus (respectively, imbedded annulus) in Z with one end an essential loop in $Z \cap P$ and the other end in another component of $Z \cap \partial M'$ is homotopic (respectively, isotopic) in Z, keeping its ends in $Z \cap \partial M'$, into one of the C_j.

Now let C be the union of all the selected C_j. Let X be the union of $\bigcup V_i$ with a small regular neighborhood of C in M'. That is, X is obtained from $\bigcup V_i$ by adding product neighborhoods of the C_j, each meeting a component of the frontier of $\bigcup V_i$ in an annulus and meeting ∂M in an incompressible annulus contained in the interior of P'. We call X the *pre-refined compression submanifold.* Each component of X is a compression body, indeed X deformation retracts to $\bigcup V_i$. Also, the frontier of X is incompressible in M. For since the annuli $Z \cap (\bigcup V_i)$ are incompressible in the frontier of $\bigcup V_i$, a compression of the frontier of X would lead to a compression of the frontier of $\bigcup V_i$.

5. The enclosing properties

We will now give the first enclosing property of the pre-refined compression submanifold X.

Proposition 5.1. *Let A be an admissible singular annulus in (M, P) having one end in $\overline{\partial M - P}$ and the other an essential loop in P. Then A is* *admissibly homotopic into X. If A is a collection of disjoint imbedded annuli, each having one end in $\overline{\partial M - P}$ and the other an essential loop in P, then A is admissibly isotopic into X.*

Proof. We will first prove the case when A is an admissible singular annulus. For a union of a disjoint collection, the argument is similar, and we will just indicate the necessary modifications.

Recall the union of normally imbedded relative compression body neighborhoods $\bigcup V_i$ used in the construction of X. The normal core (M', P') is $\overline{M - \bigcup V_i}$. Let S be the frontier of $\bigcup V_i$.

Here is a sketch of the proof. We first treat the case when one boundary component of A maps into a component of P that is disjoint from M'. In this case A can be moved entirely into $\bigcup V_i$. In the remaining cases, we may assume that one boundary component of A maps into P', and hence that the preimage of the frontier S of $\bigcup V_i$ consists of circles (not arcs). When a pair of these circles bounds a subannulus of A that maps into $\bigcup V_i$, there is an admissible homotopy that pulls that subannulus out of $\bigcup V_i$. After carrying out these homotopies, the preimage of S consists of a single circle, dividing A into an annulus A_1 that maps to $\bigcup V_i$ and an annulus A_2 that maps to M'. If A_2 is inessential in (M', P'), there is an admissible isotopy that moves A into $\bigcup V_i$. Otherwise, lemma 2.4 shows that A_2 can be moved onto one of the selected annuli C_k used in the construction of X, so again A can be moved into X.

Notice first that each component of P' is a deformation retract of a component of P. On the other hand, by construction of the normally imbedded compression bodies, a component of P disjoint from P' must be of the form $G \times \{0\} \cup \partial G \times \mathrm{I}$ for some constituent G of some V_j.

Let A be a singular annulus as in the proposition. Since each component of P' is a deformation retract of a component of P, we may change A by admissible homotopy to assume that the other boundary component maps either to P' or to a component P_0 of P that is disjoint from P'. Suppose the latter. From above, P_0 is of the form $G \times \{0\} \cup \partial G \times \mathrm{I}$ for some constituent G of some V_j. There is a collection of properly imbedded discs in V_j whose union separates P_0 from S. Put A transverse to this union. Since A is incompressible, all circles in the preimage of the discs must be inessential in A, so using irreducibility they can be removed from the preimage by admissible homotopy of A. After this is completed, A maps entirely into V_j, and the proposition is verified. So we may assume that one boundary component of A maps to P'.

Put A transverse to S. Since one boundary component maps to $\overline{\partial M - P}$ and the other to P', the preimage of S is disjoint from ∂A, so consists of circles. Since S is incompressible, any circles in the preimage that are contractible in A can be removed by admissible homotopy, so we may assume that each circle in the preimage is essential in A and in S. Suppose there is an annulus A' in A which lies between two adjacent circles of the preimage, such that A' maps to some V_j. Let E be a union of cocore 2-discs for the 1-handles of V_j. Since no boundary points of A' map to E, the preimage of E in A' consists of circles, and since $\partial A'$ is essential in S these circles are contractible in A' and may be removed by homotopy of the map on A' relative to $\partial A'$. Therefore there is an admissible homotopy of A that moves A' off of E, and then out of $\bigcup V_i$, eliminating two circles in the preimage of S. Eliminate all such annuli A'. Then, since one boundary component of A maps to $\overline{\partial M - P}$ and the other to P', the preimage of S is exactly one circle, which separates A into two annuli A_1 and A_2 mapping to $\bigcup V_i$ and M' respectively.

Suppose first that A_2 is inessential in (M', P'). This implies that A_2 is admissibly homotopic relative to $A_2 \cap S$ into $\bigcup V_i$. Consequently, A is admissibly homotopic into $\bigcup V_i$. On the other hand, if A_2 is essential then by lemma 2.4, there is an admissible homotopy of A_2, and hence of A, that moves A_2 into a Seifert-fibered component of Σ, and then onto one of the annuli C_j. After such a homotopy, A lies in X, completing the proof for the case of a singular annulus.

If A is imbedded, or is a disjoint collection of imbedded annuli, the previous arguments can be carried out using isotopies instead of homotopies; one must use lemma 4.2 of [6] when removing inessential annuli, and lemma 1.1 to obtain isotopies moving the annuli A' out of $\bigcup V_i$. □

Here is the second enclosing property of the pre-refined compression submanifold X.

Proposition 5.2. *If S is a component of the frontier of X, and γ is an essential closed curve in S which is homotopic in M into P, then γ is homotopic in S into $X \cap P$.*

As remarked in the introduction to this section, this is not stated in the the usual form of an enclosing property, but it is equivalent to the assertion that any singular annulus with one end an essential loop γ in S and the other end in P is homotopic, relative to γ and keeping the other end in P, into S. For suppose that such a singular annulus is given, and that the condition in proposition 5.2 holds. The homotopy in S gives a singular annulus in S with one end equal to γ and the other in $S \cap P$. The two annuli glue together to give an essential annulus with both ends in P. Since this is homotopic into P, the original annulus is homotopic relative to γ and keeping its other end in P to the annulus in S. On the other hand, suppose that the enclosing property holds and that γ is homotopic in M into P. The homotopy gives a singular annulus, and when deformed into S, relative to γ, it becomes a homotopy in S into $S \cap P$.

Proof. Let V be the component of X that contains S, and let $Q = V \cap P$. Since V is a compression body, γ is homotopic in V to a loop γ_1 in a free side of (M, P).

Proposition 5.1 implies that γ_1 and hence γ are homotopic in V into Q. Since γ is essential, we may assume that the homotopy from γ into Q misses an open regular neighborhood of a collection of cocores for the 1-handles of V. The complement in V of this open regular neighborhood has a structure as a product $G \times \mathrm{I}$ where S and Q are contained in $G \times \{0\}$. Projecting the homotopy to $G \times \{0\}$ gives a homotopy in $G \times \{0\}$ from γ to a loop in Q. This implies that γ is homotopic in S into Q. (The homotopy is a map from an annulus into $G \times \{0\}$. Put it transverse to ∂S and then deform it to remove contractible simple closed curves from the preimage of ∂S. Then, the domain contains a smaller annulus with one boundary component mapping as γ, the other mapping to ∂S, and the interior having image disjoint from ∂S, and hence contained in S.) The restriction of the homotopy to this subannulus is a homotopy in S carrying γ into Q. $\square$

Let Z be a codimension-zero submanifold of M. A component R of $\overline{M-Z}$ which is a product of the form $S \times \mathrm{I}$ where $R \cap Z = S \times \{0\}$ is called an *adherent* component of $\overline{M-Z}$. We also say it is an adherent component *for* Z. An adherent component for a component of the pre-refined compression submanifold X can contain a free face of (M,P) (and hence can contain other components of X). The next proposition shows that this phenomenon can occur only in a very restricted way.

Proposition 5.3. *Let R be an adherent component for a component W_0 of X. Then R can be given product coordinates $S \times I$ so that $R \cap W_0 = S \times \{0\}$, $R \cap P = \partial S \times I$, and $R \cap X = S \times \{0\} \cup S \times [1/2, 1]$.*

Proof. Write $R = S \times \mathrm{I}$ with $S \times \{0\} = R \cap W_0$. From the construction of X, we may assume that $S \times \partial\mathrm{I} \subset P$. Any essential loop in $S \times \{1\} \cap P$ is homotopic into $S \times \{0\}$, and consequently homotopic into the free side of (M,P) contained in W_0. By proposition 5.1, the singular annulus determined by this homotopy is admissibly homotopic into X and hence into W_0. Thus every loop in $S \times \{1\} \cap P$ is homotopic in P into $\partial S \times \{1\}$. It follows that each component of $S \times \{1\} \cap P$ is an annulus containing a boundary component of S. If S itself were an annulus, then either one of the annuli C_i would be inessential in (M', P'), or two of the annuli C_i and C_j would be parallel in (M', P'), neither of which is permitted in our construction. So each component of $S \times \{1\} \cap P$ is a collar neighborhood of a boundary circle of $S \times \{1\}$, and the product structure can be reselected so that $R \cap P = \partial S \cap \mathrm{I}$. The component of X that contains the free face $S \times \{1\}$ of (M,P) has incompressible frontier and is disjoint from $S \times \{0\}$. Therefore its frontier is admissibly isotopic to $S \times \{1/2\}$, so the component is admissibly isotopic to $S \times [1/2, 1]$. $\square$

To construct the refined relative compression body neighborhood (W,Q) of a free side F, we start with the component W_0 of X that contains F, add its adherent components, and put $Q = W \cap P$. For each component R of $\overline{M-W}$ such that $R \cap W$ is connected, $\pi_1(R \cap W) \to \pi_1(R)$ is not surjective, since otherwise R would be a product with one end a component of the frontier of W (by theorem 10.5 of [4]), and R would have been adherent. The pared properties of (M, P) imply that no component of the frontier of W is a torus or annulus.

The enclosing properties of the refined relative compression body neighborhood are immediate consequences of the corresponding properties of X.

Proposition 5.4. *Let (W, Q) be a refined relative compression body neighborhood of the free side F of (M, P), and let A be any admissible singular annulus in (M, P) having one end in F and the other an essential loop in P. Then A is admissibly homotopic into W. If A is a collection of disjoint imbedded annuli, each having one end in F and the other an essential loop in P, then A is admissibly isotopic into W.*

Proof. By proposition 5.1, A is admissibly homotopic (or isotopic, in the latter case) into X. Since W contains the component of X that contains F, A will then lie in W. □

Proposition 5.5. *Let (W, Q) be a refined relative compression body neighborhood of the free side F of (M, P). If S is a component of the frontier of W, and γ is an essential closed curve in S which is homotopic in M into P, then γ is homotopic in S into Q.*

Proof. If S is a component of the frontier of W, then it is a component of the frontier of X, so proposition 5.2 applies. □

6. Existence and uniqueness

Our uniqueness result gives a characterization of the refined relative compression body neighborhood, in terms of the second enclosing property.

Theorem 6.1. *Let F be a free side of a pared 3-manifold (M, P), and let (W, Q) be the refined relative compression body neighborhood of F. Suppose that (W', Q') is a relative compression body neighborhood of F in (M, P) such that:*

(i) *$W' \cap P = Q'$, and if G is a free face of (W', Q'), then either $G \subset \partial M$ or $G \cap \partial M = \partial G$.*
(ii) *If R is a component of $\overline{M - W'}$ such that $R \cap W'$ is connected, then $\pi_1(R \cap W') \to \pi_1(R)$ is not surjective.*
(iii) *If S' is a component of the frontier of W', and γ is an essential loop in S' which is homotopic in M into P, then γ is homotopic in S' into Q'.*

Then (W', Q') is admissibly isotopic to (W, Q).

Proof. Recall that W was constructed starting from a normally imbedded relative compression body neighborhood V of F and a regular neighborhood $N(A)$ of a union A of disjoint annuli in M', forming the union W_0 of these, then adding in the adherent components of the complement. By lemma 1.3, and by use of condition (ii), we may assume that V lies in the topological interior of W'. Using lemma 1.1, we may assume that each properly-imbedded constituent of V equals $G \times \{1/2\}$ for some constituent G of W'.

We will show that A is admissibly isotopic, fixing $A \cap V$, into W'. Let T' denote the frontier of W'. Note that by condition (i), $\partial T' \subset P$. We may assume that A is transverse to T', and since A is incompressible, that none of the components of the intersection is a contractible circle. Consider an arc of intersection α. Since V lies in the interior of W', α must separate off a disc E in A with boundary consisting of α and an arc α'' in P. We may choose E outermost, so that it meets T' only in α.

We claim that the endpoints of α lie in the same component of $\partial T'$. Suppose not, so that they lie in distinct boundary components C_1 and C_2. Since α'' lies in P, C_1 and C_2 lie in the same component P_0 of P, and since this component is an annulus or a torus, we may orient C_1 and C_2 to represent the same element of $\pi_1(P_0)$. Since α is homotopic to α'', the loop $\alpha * C_2 * \overline{\alpha} * \overline{C_1}$ is contractible in M and hence in T', so the component T of T' that contains $C_1 \cup \alpha \cup C_2$ is an annulus. Since its boundary components lie in P_0, and (M, P) is pared, T is homotopic into P_0. This implies that the component of $\overline{M - W'}$ that contains T is adherent, contradicting condition (ii) and establishing the claim.

Choose an arc α' in $\partial T'$ connecting the endpoints of α. The existence of E shows that $\alpha \cup \alpha'$ is homotopic into P. By condition (iii), $\alpha \cup \alpha'$ is homotopic in T' into $\partial T'$, which implies that α is parallel in T' into $\partial T'$. So there is a disc E' in T' with boundary the union of α and an arc in P. Since P is incompressible, the disc $E \cup E'$ is parallel into P. So there is an admissible isotopy of A that moves E across the region of parallelism and through E', eliminating α and possibly other arcs of $A \cap T'$. Repeating this process, we may eliminate all arcs of $A \cap T'$.

Now let β be a circle of $A \cap T'$ that is outermost on A, that is, so that the interior of the annulus L on A between β and a circle of $A \cap P$ has no intersections with T'. By condition (iii) for W', β and some boundary component of T' cobound an annulus L' in T'. Since $\overline{M - V}$ is pared, $L \cup L'$ is parallel in $\overline{M - V}$ into P. So there is an admissible isotopy of A in $\overline{M - V}$ that moves L through the region of parallelism and across L', eliminating β and possibly other circles of $A \cap T'$. Repeating this process, we we produce an admissible ambient isotopy that fixes V and moves A to be disjoint from T', and hence contained in W'. The reverse of this isotopy fixes moves W' so that it contains the original A. By a further isotopy, we may assume that W' contains W_0 in its topological interior.

Now consider a component S of the frontier of W_0, and let R be the component of $\overline{M - W_0}$ that contains S. Since W lies in the topological interior of W', there is a component R' of $\overline{W' - W}$ that contains S. Suppose first that R is adherent, so $R \subset W$ and R can be given product coordinates $S \times \mathrm{I}$ so that $R \cap W_0 = S \times \{0\}$ and $R \cap P = \partial S \times \mathrm{I}$. Any components of the frontier of W' contained in R would be parallel into $R \cap \partial M$, contradicting condition (ii). So in this case, $R' = R$. Suppose now that R is not adherent, so that S is a component of the frontier of W. Since $\pi_1(F) \to \pi_1(W')$ is surjective, we must have $R' \cap W_0 = S$ and $\pi_1(S) \to \pi_1(R')$ surjective. So R' is a product $S \times \mathrm{I}$ with $S = S \times \{0\}$. Now let α be any essential loop in $Q' \cap R'$. Since W' is a compression body, there is a singular annulus with one end equal to α and the other in F. By proposition 5.4, this annulus is admissibly homotopic into W, thus every essential loop in $Q' \cap R$ is homotopic into S. So each component of $Q' \cap R$ is an annulus meeting S in one or both boundary components. If one of them meets S in both boundary components, then since R is a product with one end equal to S, S is parallel into P and W violates condition (ii). So each annulus has exactly one boundary component in S, that is, $Q' \cap \partial R$ is a collar neighborhood of ∂S. If G is the free side of (W', Q') in R, then G cannot be in ∂M, since R was not adherent. So by condition (i), G is properly imbedded. Therefore there is an admissible isotopy of W that expands it to include R'. Repeating for all components of the frontier of W_0, we move W onto W'. □

We are now going to show that a disjoint collection of refined relative compression body neighborhoods can be selected which includes all free sides of (M, P), and that such a collection is unique up to isotopy.

Theorem 6.2. *Let (M, P) be a pared manifold.*

(i) *There exists a subcollection $F_1, \ldots, F_r$ of the free sides of (M, P) and a disjoint collection $W_1, \ldots, W_r$, where each W_i is a refined relative compression body neighborhood of F_i, such that the union of the W_i contains all free sides of (M, P). The union of the W_i is unique up to admissible isotopy.*

(ii) *Each free face F which is not one of the F_i is incompressible, and either $(M, P) = (F \times I, \partial F \times I)$, so that (M, P) is the refined relative compression body neighborhood of both of its free faces, or the refined relative compression body neighborhood of F is a collar neighborhood which can be taken to lie in $\bigcup W_i$.*

(iii) *If A is a singular annulus with one end contained in a free side of (M, P) and the other end an essential loop in P, then A is admissibly homotopic into $\bigcup W_i$. A union of disjoint imbedded annuli, each having one end in $\overline{\partial M - P}$ and the other end an essential loop in P, is admissibly isotopic into $\bigcup W_i$.*

(iv) *If γ is an essential loop in the frontier of $\bigcup W_i$ and γ is homotopic in M into P, then γ is homotopic in the frontier of $\bigcup W_i$ into P.*

Proof. Proposition 5.3 shows that if we add to X all adherent components of the complements of its components, then the result is a disjoint union $\bigcup W_i$ of refined relative compression body neighborhoods of a subcollection of the free sides of X, and this union contains all free sides. It shows moreover that the remaining free sides are as described in statement (ii).

For the uniqueness in (i), one may adapt the proof of theorem 6.1, but instead we will argue by induction on the number of W_i that are not collar neighborhoods of a free face of (M, P). If all are collars, then the uniqueness is simply the uniqueness of collar neighborhoods. For the induction, suppose that W_1 is not a collar neighborhood of a free face, and that that $W_1', \ldots, W_s'$ is a second collection as in (i). By theorem 6.1, we may assume that $W_1 = W_1'$. Let $(M_1, P_1) = (\overline{M - W_1}, \overline{M - W_1} \cap P)$, which has the constituents $G_1, \ldots, G_n$ of W_1 among its free faces. Notice that since W_1 satisfied the hypotheses of theorem 6.1, any collar neighborhood of a G_k in M_1 satisfies these same hypotheses, so is a refined relative compression body neighborhood of G_k in M_1. The W_i and W_j' might not be refined relative compression body neighborhoods in (M_1, P_1); they satisfy conditions (i) and (iii) of theorem 6.1, but might fail condition (ii) since adherent components containing some of the G_k might be created when W_1 is removed. Let Y_i and Y_i' be obtained from W_i and W_i' by adding in any such adherent components. For any G_k not contained in one of the Y_i, add another Y_ℓ which is a collar neighborhood of G_k in M_1, and similarly for each G_k not contained in one of the Y_j'. By induction, the union of the Y_i is admissibly ambiently isotopic in (M_1, P_1) to the union of the Y_i', so we assume that the unions are equal. This shows that the Y_ℓ that are collar neighborhoods of G_k's correspond to the Y_ℓ' that are collar neighborhoods of G_k's. The remaining G_k must lie in adherent components of both the W_i and the W_i'; by uniqueness of collar neighborhoods, we may assume that these adherent components are equal. Then, the union of the W_i for $i \geq 2$ equals the union of the W_i' for $i \geq 2$ and the uniqueness is established.

Now $\bigcup W_i$ contains a refined relative compression body neighborhood of each free face of (M, P), either one of the W_i or a collar neighborhood. Consequently, it contains a pre-refined compression submanifold X. Statement (iii) follows immediately from proposition 5.1. Statement (iv) follows directly from theorem 6.1. □

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