H^p SPACES

CHRISTIAN REMLING

1. Fourier series

We denote the unit disk and unit circle in the complex plane by $D = \{z \in \mathbb{C} : |z| < 1\}$ and $T = \{z : |z| = 1\}$, respectively. (T as in *torus,* because that and not S^d is the proper object in higher dimensions in this context.) T can be naturally identified with an interval of length 2π, say $I = [0, 2\pi)$, by mapping $I \to T$, $x \mapsto e^{ix}$. (You are perhaps used to the theory being developed on I, but T is slightly preferable here because its topology is more useful. This topology comes into play for example when we talk about continuity of functions $f : T \to \mathbb{C}$.) In particular, we can use such a map to move Lebesgue measure over to T. We also normalize and give the full circle measure 1. We will denote this (Borel) measure on T by σ, so

$$\sigma(\{e^{ix} : a < x < b\}) = \frac{b-a}{2\pi}, \qquad 0 \le a < b \le 2\pi.$$

In other words, σ is the normalized arc length measure.

Recall next that the space $L^p(T)$ (with the measure σ understood, but not indicated in the notation) consists of the Borel measurable functions $f : T \to \mathbb{C}$ with $\int_T |f|^p \, d\sigma < \infty$ if $1 \le p < \infty$, and $L^\infty(T)$ contains the essentially bounded measurable functions. In both cases, functions that agree almost everywhere are usually identified.

Exercise 1.1. Show that $L^p(T) \subseteq L^q(T)$ if $p \ge q$ and that $C(T) \subseteq L^p(T)$ for all $1 \le p \le \infty$.

For $f \in L^1(T)$, we define its *Fourier coefficients* as

$$a_n(f) = \int_T f(z) z^{-n} \, d\sigma(z), \qquad n \in \mathbb{Z}. \tag{1.1}$$

If we make use of the correspondence $I \to T$ that we just discussed, then this takes the more familiar form

$$a_n = \int_0^{2\pi} f(e^{ix}) e^{-inx} \, \frac{dx}{2\pi}.$$

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We will also often, somewhat inconsistently from a formal point of view but conveniently, mix these two forms and write things like $a_n = \int_T f(e^{ix})e^{-inx}\, d\sigma(x)$. The notation $\widehat{f}_n = a_n(f)$ is also common.

The cleanest theory of the Fourier transform is obtained on $L^2(T)$. This is a Hilbert space, with scalar product

$$\langle f, g\rangle = \int \overline{f} g\, d\sigma. \tag{1.2}$$

It's now easy to check that the exponentials $e_n(x) = e^{inx}$, $n \in \mathbb{Z}$ form an orthonormal system (ONS), that is,

$$\langle e_m, e_n\rangle = \delta_{mn}.$$

Exercise 1.2. Prove this.

This already implies *Bessel's inequality:* if $f \in L^2(T)$, then

$$\sum_{n\in\mathbb{Z}} |a_n(f)|^2 \le \|f\|_2^2.$$

In particular, $a_n \in \ell^2(\mathbb{Z})$ if $f \in L^2$. But we will see in a moment that this actually holds with equality.

In fact, it can be shown that $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis (ONB) of $L^2(T)$, which means, in addition to being an ONS, that the closed linear span of these functions is the whole space $L^2(T)$. The details of the argument (many different ones are possible, in fact) are not important for us, so I don't want to discuss it here. It follows that any $f \in L^2(T)$ can be expanded in terms of the e_n, and the expansion coefficients are given by $\langle e_n, f\rangle$, but this is $a_n(f)$, by comparing (1.1) with (1.2) (and recall that $z^{-1} = \overline{z}$ for $z \in T$). It also follows that we can, conversely, start out with desired expansion coefficients (as long as they are consistent with Bessel's inequality) and then there will be a corresponding function. (I obtain these conclusions by applying Hilbert space tools to the case at hand, so if you are not familiar with this material, these steps will not be clear. In this case, please try to read up on this; for example, take a look at Chapter 5 of my Functional analysis lecture notes, which are available on my homepage.) We summarize:

Theorem 1.1. *If $f \in L^2(T)$, then (Parseval's identity)*

$$\sum_{n\in\mathbb{Z}} |a_n(f)|^2 = \|f\|_2^2,$$

and f has the expansion

$$f(e^{ix}) = \sum_{n\in\mathbb{Z}} a_n e^{inx}. \tag{1.3}$$

Conversely, if $\sum |b_n|^2 < \infty$*, then there is a unique* $f \in L^2(T)$ *with* $a_n(f) = b_n$*.*

Another way of saying this is to point out that the Fourier transform $f \mapsto (a_n)_{n\in\mathbb{Z}}$ is a unitary map from $L^2(T)$ onto $\ell^2(\mathbb{Z})$.

The *Fourier inversion* formula (1.3) needs careful interpretation in our current context. The convergence takes place in $L^2(T)$. In other words, if $S_N(x) = \sum_{|n|\le N} a_n e^{inx}$ denotes a (symmetric) partial sum, then what we have is $\|S_N - f\|_2 \to 0$ as $N \to \infty$. This, by itself, does not imply that also $S_N(x) \to f(e^{ix})$ pointwise for almost every x (it does follow that this will hold after passing to a suitable subsequence $N_j \to \infty$), and in fact this question, *does the Fourier series of an* $L^2(T)$ *function converge pointwise a.e.?,* was open for a long time until it was finally answered by Carleson, in the affirmative, in 1966.

This completely clarifies (1.3) for $f \in L^2(T)$. Note that this case is already a little delicate in the sense that absolute convergence of (1.3) is not guaranteed; that would correspond to the stronger (than $a \in \ell^2$) property $a \in \ell^1$. So we already have to rely on partial cancellations due to oscillations. There are other surprises in store. For example, Kolmogorov constructed a function $f \in L^1(T)$ (and necessarily $f \notin L^2$) for which the *Fourier series* from (1.3) diverges for all x.

For $f \in L^2$, we now know that exactly the square summable sequences $a_n \in \ell^2$ are possible as Fourier coefficients. What about the Fourier coefficients of an $f \in L^1(T)$? There is no good answer to this, but one fundamental and easy general result is:

Theorem 1.2 (Riemann-Lebesgue lemma)**.** *Let* $f \in L^1(T)$*. Then* $a_n \in \ell^\infty$ *with* $\|a\|_\infty \le \|f\|_1$*, and* $a_n(f) \to 0$ *as* $n \to \pm\infty$*.*

In other words, the Fourier transform maps L^1 into c_0 (the space of sequences that converge to zero), but no claim is being made that the map is also onto, and in fact this is false.

Proof. It is obvious from the definition of a_n that $|a_n| \le \int |f|\, d\sigma = \|f\|_1$.

We not only have $L^2(T) \subseteq L^1(T)$, but it is also true L^2 is dense in L^1 (with respect to $\|\cdot\|_1$). (A very useful fact that should definitely be in your toolkit is that much smaller spaces of very nice functions, for example $C^\infty(T)$, are already dense in L^p for any $p < \infty$.) Thus, given any $\epsilon > 0$, we can pick a $g \in L^2(T)$ with $\|g - f\|_1 < \epsilon$. By the already established first part of the theorem, this implies that

$$|a_n(g) - a_n(f)| = |a_n(g - f)| < \epsilon.$$

Moreover, $a_n(g) \in \ell^2$, by Theorem 1.1, so in particular $a_n(g) \to 0$ as $n \to \pm\infty$, and thus we can find an $N \geq 1$ such that $|a_n(g)| < \epsilon$ for all $|n| \geq N$. Putting things together, we then see that also $|a_n(f)| < 2\epsilon$ for $|n| \geq N$, as required. □

Exercise 1.3. Give an easy direct proof that L^2 is dense in L^1, by considering $g_n = \chi_{\{|f|\leq n\}} f$ for a given $f \in L^1$.

I now want to do a very quick review, without proofs mostly, of convolutions. The *convolution* of two functions $f, g \in L^1(T)$ is defined as

$$(f * g)(e^{ix}) = \int_T f(e^{it}) g(e^{i(x-t)})\, d\sigma(t) \tag{1.4}$$

(you are probably familiar with this in the neater looking version $(f * g)(x) = \int f(t)g(x-t)\, dt$ on the real line rather than T). This can indeed be defined for $f, g \in L^1$, for almost every x, and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. Both these claims follow from Fubini-Tonelli, by (formally, at first) integrating the formula for $f*g$ and then doing the x integral first. More generally, if $f \in L^1$ and $g \in L^p$, then $f*g \in L^p$ and $\|f*g\|_p \leq \|f\|_1 \|g\|_p$.

Exercise 1.4. Prove that the convolution product is commutative, $f * g = g * f$, and, if not bored by this, you could also verify that it in fact has all the algebraic properties of a (ring) product. For example, $f * (g + h) = f * g + f * h$.

Convolutions are important mainly for two reasons. First of all, the smoothness of a function can be improved by convolving with a nice function. The general principle is that the convolution is always at least as nice as the nicer of the two factors. A precise statement along these lines is that if $f \in L^1$ and $g \in C^k$, then $f * g \in C^k$. Second, convolving with a narrow high function of integral 1 produces an approximation to the original function. To formulate a rather general precise statement along these lines, we first introduce the notion of an *approximate identity:* a sequence of functions $k_n \in L^1(T)$ (or it could also be a family of functions indexed by a real parameter rather than $n \in \mathbb{N}$, but I'll focus on this case for now) is called an approximate identity if $\int_T k_n\, d\sigma = 1$, $\|k_n\|_1 \leq C$, and

$$\lim_{n\to\infty} \int_{|x|>\delta} |k_n(e^{ix})|\, d\sigma(x) = 0 \tag{1.5}$$

for every $\delta > 0$ (and I now somewhat inconsistently imagine T being parametrized by $-\pi \leq x < \pi$, say). In other words, k_n has integral 1, and almost all of this is concentrated near $x = 0$ once n gets large.

For a concrete example, we could take $k_n(x) = 2n$ for $|x| < 1/n$ and $k_n(x) = 0$ otherwise.

Theorem 1.3. *Let $k_n \in L^1(T)$ be an approximate identity. If $f \in L^p(T)$, $1 \le p < \infty$, then $\|k_n * f - f\|_p \to 0$. Similarly, if $f \in C(T)$, then $(k_n * f)(x) \to f(x)$ uniformly on $x \in T$.*

So we now have a very explicit version of the general fact I mentioned above, in the proof of Theorem 1.2: arbitrary L^p functions can be approximated by nice functions. Namely, we can fix an approximate identity k_n consisting of such nice functions, and why not go all the way and take $k_n \in C^\infty(T)$, and then we form $k_n * f$. These functions are smooth, and they converge to f (in what sense exactly will depend on the properties of f).

Theorem 1.3 also explains the terminology: an approximate identity almost acts like a multiplicative identity when convolution products with it are taken.

I don't want to give a full *proof* of Theorem 1.3 here, I will only briefly discuss the easiest case, when $f \in C(T)$. But you should at least understand intuitively why $k_n * f = f * k_n$ is close to f: if you take another look at the definition (1.4), you'll see that $(f * k_n)(e^{ix})$ can be thought of as an average of values $f(e^{it})$, and mostly these t's are taken from a small neighborhood of x, because of (1.5). So if f has some modest regularity (as all L^1 functions do, by Lebesgue's differentiation theorem), then we may expect this to be close to $f(e^{ix})$.

Let's now do the formal argument for $f \in C(T)$. We can write

$$(k_n * f)(e^{ix}) - f(e^{ix}) = \int_T k_n(e^{it})(f(e^{i(x-t)}) - f(e^{ix}))\, d\sigma(t) \tag{1.6}$$

(since $\int k_n = 1$). Now $f \in C(T)$ is uniformly continuous on the compact space T, so, given $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(e^{i(x-t)}) - f(e^{ix})| < \epsilon$ for all x, provided that $|t| < \delta$. So the contributions coming from $|t| < \delta$ to the integral from (1.6) are (in absolute value) $< \epsilon\|k_n\|_1 \le C\epsilon$. On the other hand, since $f \in C(T)$ is bounded (again, because T is compact), the contributions from $|t| > \delta$ are $\le 2\|f\|_\infty \int_{|t|>\delta} |k_n|\, d\sigma$, which goes to zero as $n \to \infty$.

The convolution also interacts nicely with the Fourier transform.

Theorem 1.4. *Let $f, g \in L^1(T)$. Then $a_n(f * g) = a_n(f)a_n(g)$.*

Exercise 1.5. Prove this (by an easy direct calculation).

Theorem 1.5 (Uniqueness)**.** *Let $f, g \in L^1(T)$ and suppose that $a_n(f) = a_n(g)$. Then $f = g$.*

The interesting aspect of this is that recovery of a function from its Fourier coefficients will always work (in principle, at least), even though (1.3) may be inadequate to get the job done.

Proof. By considering $f - g$, we see that it suffices to prove that $f = 0$ if $a_n(f) = 0$. This is already clear, by Theorem 1.1, if $f \in L^2(T)$. In general, if we only have $f \in L^1$, fix an approximate identity with the additional property $k_j \in L^2$ (for example, use the rectangular functions given above) and consider $k_j * f \in L^2$. By Theorem 1.4, $a_n(k_j * f) = a_n(k_j)a_n(f) = 0$, so $k_j * f = 0$ by what we just discussed. However, $k_j * f \to f$ in L^1 by Theorem 1.3, so $f = 0$ as well. □

Corollary 1.6 (Fourier inversion)**.** *Let $f \in L^1(T)$, and suppose that $a_n(f) \in \ell^1$ (that is, $\sum |a_n| < \infty$). Then in fact $f \in C(T)$ and $f(e^{ix}) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ for all x.*

It would be more careful to say here that $f \in L^1$ is almost everywhere equal to a continuous function. It is clear that more than this cannot be true because the Fourier coefficients $a_n(f)$ are computed by an integral, so are insensitive to a change of values of f on a null set. Also, I already stated that we identify functions that agree almost everywhere, so there should be no real danger of confusion.

Proof. Since $\ell^1 \subseteq \ell^2$, Theorem 1.1 shows that $f \in L^2$ and (1.3) holds (with convergence in L^2). As we already reviewed above, convergence in L^2 implies pointwise convergence almost everywhere on a subsequence, so

$$f(e^{ix}) = \lim_{j\to\infty} \sum_{n=-N_j}^{N_j} a_n e^{inx}$$

for almost every x. But actually the passage to a subsequence was unnecessary since the series itself converges by our assumption that $a \in \ell^1$. (Or we could have applied Carleson's theorem, but that's cracking a nut with a sledgehammer.) Moreover, the limit is uniform in x, for the same reason, since

$$\left| f(e^{ix}) - \sum_{n=-N}^{N} a_n e^{inx} \right| \le \sum_{|n|>N} |a_n| \to 0 \qquad (N \to \infty).$$

The finite sums are continuous functions of x, so $f \in C(T)$. □

If this proof is read superficially, one could get the impression that all this could have been done immediately following Theorem 1.1. But that's not the case: we do use uniqueness (Theorem 1.5) here, in the very first step, when we conclude that $f \in L^2$ from the fact that $a \in \ell^2$.

This only works because we now know that the Fourier coefficients determine the function for a general $f \in L^1$. Theorem 1.1 only tells me that there is a unique $f \in L^2$ with given Fourier coefficients $a_n \in \ell^2$, but it does not rule out the possibility that there might be other functions $g \in L^1 \setminus L^2$ with those same Fourier coefficients. This only follows from Theorem 1.5.

2. The Poisson kernel

The *Poisson kernel* is the family of functions $P_r \in C(T)$, $0 \le r < 1$, given by

$$P_r(e^{ix}) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}. \tag{2.1}$$

The series converges uniformly in x for any given $r < 1$. In fact, it is a geometric series (or, more precisely, a sum of two of these, one for $n < 0$ and one for $n > 0$), and we can evaluate it explicitly. We find

$$P_r(e^{ix}) = \frac{1 - r^2}{1 - 2r\cos x + r^2}. \tag{2.2}$$

Exercise 2.1. Carry out this calculation in more detail.

From the uniform convergence of (2.1) and the orthogonality of the exponentials we see that

$$a_n(P_r) = \int_T P_r(e^{ix}) e^{-inx}\, d\sigma = |r|^n.$$

Thus, by Theorem 1.4, $a_n(P_r * f) = |r|^n a_n(f)$ for any $f \in L^1(T)$, and these exponentially decaying Fourier coefficients satisfy the assumptions of Corollary 1.6. Hence

$$(P_r * f)(e^{ix}) = \sum_{n=-\infty}^{\infty} a_n(f) |r|^n e^{inx}, \tag{2.3}$$

and this series converges absolutely and uniformly in x.

On the other hand, P_r is an approximate identity when $r \to 1-$:

$$\int_T P_r\, d\sigma = a_0(P_r) = 1,$$

and also $\int |P_r|\, d\sigma = 1$, since $P_r \ge 0$. Moreover, if $|x| > \delta$, then $\cos x$ is bounded away from 1, say $\cos x \le 1 - \eta$ (recall that we can let x vary over $[-\pi, \pi)$, say, so that then $x = 0$ is the only point in our interval where $\cos x = 1$). This will guarantee that the denominator of (2.2) is bounded away from zero, so $P_r(e^{ix}) \le C(1 - r^2)$ on $|x| > \delta$, and this gives us (1.5), with P_r taking the role of k_n, and the limit $n \to \infty$

replaced by $r \to 1-$. This was the final requirement on an approximate identity.

When this is combined with (2.3), we obtain what is usually called a *summation method* for Fourier series, that is, a method for recovering $f \in L^1(T)$ from $\sum a_n e^{inx}$ even though this series may well be divergent: we work with $\sum a_n |r|^n e^{inx}$ instead, which is guaranteed to converge, and we then also know, from Theorem 1.3, that these modified series converge to f in $L^1(T)$ as $r \to 1-$. This particular method of assigning a value to a potentially divergent series is also called *Abel summation.* So we can now summarize and say that while the original Fourier series may not recover the function, due to its possible divergence, its Abel summed version always will, at least if we interpret things carefully (the convergence will take place in L^1 when we send $r \to 1-$).

If $a \in \ell^1$ and thus the issues the summation method is supposed to address were absent, then indeed the summation method won't do anything: in this case,

$$\lim_{r\to 1-} \sum_{n=-\infty}^{\infty} a_n |r|^n e^{inx} = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

for all x.

Exercise 2.2. Prove this.

If we only wanted a summation method for Fourier series, then in fact the more natural choice would have been *Cesaro summation,* which works with the averages $(1/N)\sum_{n=0}^{N-1} S_n$, with $S_n = \sum_{|k|\le n} a_k e^{ikx}$. But the Poisson kernels are important for us for another reason: they provide a link from the theory of Fourier series to complex analysis on the unit disk D. To make this connection explicit, reorganize r, x into one complex variable $z = re^{ix}$, $z \in D$. For $n \le -1$, we can write $r^{|n|}e^{inx} = \overline{z}^{|n|}$, and thus

$$P_r(e^{ix}) = 1 + \sum_{n=1}^{\infty} (z^n + \overline{z}^n) = 1 + 2\,\mathrm{Re} \sum_{n=1}^{\infty} z^n = \mathrm{Re}\, \frac{1+z}{1-z}. \tag{2.4}$$

This function $(1+z)/(1-z)$ is holomorphic on D. As a consequence, its real part $u(z) = P_r(e^{ix})$ is *harmonic*, that is, $u \in C^2(D)$ satisfies *Laplace's equation* $\Delta u = 0$, with $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$, $z = s + it$ (normally we would write $z = x+iy$, but this clashes with our previous use of x as the phase of z in polar coordinates).

Exercise 2.3. Confirm directly, by a brute force calculation, that P_r from the right-hand side of (2.2) is harmonic, in the sense explained above. More specifically, form $z = re^{ix} \in D$, then express $P_r(e^{ix})$ in terms of s, t, with $z = s + it$, and then verify that $P(s,t)$ is harmonic. (You are not going to enjoy this, and the exercise is unnecessary from a strictly logical point of view since we already proved that P is harmonic, but it will help you gain some confidence with these things.)

Now let's take another look at $P_r * f$, with $f \in L^1(T)$. I now want to interpret

$$(2.5) \qquad (P_r * f)(e^{ix}) = (f * P_r)(e^{ix}) = \int_T f(e^{it}) P_r(e^{-it}e^{ix})\, d\sigma(t)$$

also as a function of $z = re^{ix} \in D$. The function $z \mapsto P_r(e^{-it}e^{ix})$ is still harmonic for any fixed t, as we can see by slightly adapting the calculation from (2.4). This (strongly) suggests that $P * f$, thought of as a function of $z = re^{ix} \in D$, is harmonic on D. This impression is correct, though to prove it formally, we'd have to justify pulling the Laplacian under the integral sign (it's not very hard to give a proper proof, and please try to do it if interested; a slicker proof than the one sketched can be based on the mean value property of harmonic functions, if you're familiar with this). Alternatively, we can obtain the same conclusion from (2.3). We then use the fact that each of the summands $z \mapsto z^{|n|}$, $z \mapsto \overline{z}^{|n|}$ is harmonic and that the convergence is locally uniform, that is, it is uniform on each smaller disk $|z| \le R$, $R < 1$. Finally, a uniform limit of harmonic functions is harmonic.

Let me summarize.

Theorem 2.1. *Let $f \in L^p(T)$, $1 \le p < \infty$. Then $F(z) = P_r * f$ is a harmonic function on D, and $f_r \to f$ in $L^p(T)$ as $r \to 1-$, with $f_r(e^{ix}) = F(re^{ix})$, and also $f_r(e^{ix}) \to f(e^{ix})$ for almost every x.*

If $f \in C(T)$, then $F(z) = P_r * f$ is harmonic on D and has a continuous extension to $\overline{D}$, with $F(e^{ix}) = f(e^{ix})$.

We discussed (but did not always prove formally) most of this above. The convergence $f_r \to f$ in L^p follows from Theorem 1.3 since $f_r = P_r * f$ and P_r is an approximate identity. The pointwise convergence would need a separate argument, but I don't want to do this here.

If $f \in L^\infty(T)$, then we can still obtain f as the pointwise limit (almost everywhere) of the f_r; in fact, this much is obvious since $L^\infty(T) \subseteq L^1(T)$. However, f_r need not converge to f in L^∞, which is also clear, since a uniform limit of the continuous functions f_r would be continuous itself, but of course a general $f \in L^\infty(T)$ need not be continuous anywhere.

Essentially, we have obtained a rather neat complex analytic reinterpretation of Abel summation: Given an $f \in L^1(T)$, we associate with it a harmonic function F on D, which has f as its boundary values $f(e^{ix}) = \lim_{r\to 1-} F(re^{ix})$. Along a circle $|z| = r$, the function $F(re^{ix}) = P_r * f$ is the modified Fourier series from (2.3).

The *Dirichlet problem* on the disk is the boundary value problem

$$\Delta u = 0, \qquad u = f \text{ on } T; \tag{2.6}$$

more precisely, $f \in C(T)$ is given, and we are looking for a function $u \in C(\overline{D})$ that satisfies $\Delta u = 0$ on D and agrees with f on T. The last part of Theorem 2.1 says that we can solve the Dirichlet problem explicitly by the *Poisson integral* $u = P_r * f$.

The solution to (2.6) is unique, and this follows conveniently from the *maximum principle* for harmonic functions: if a harmonic function $u : D \to \mathbb{R}$ has a local maximum (or minimum) at some point $z_0 \in D$, then u is constant (and this is a quick consequence of the mean value property of harmonic functions, which I already alluded to above). So if u, v both solve (2.6), then $w = u - v$ is harmonic on D and $w = 0$ on T. Then both $\text{Re}\, w$ and $\text{Im}\, w$ have the same properties, and these functions must have an extremum on D since they are constant $(= 0)$ on T, so w is constant. Since $w = 0$ on T, it follows that $w = 0$ everywhere, so $u = v$, as claimed.

It is now natural to ask if all this can be turned around. Namely, can I start out with a harmonic function $F : D \to \mathbb{C}$ and then expect this to have boundary values $f(e^{ix}) := \lim_{r\to 1-} F(re^{ix})$? Moreover, if that works, will then $F = P_r * f$ turn out to be the Poisson integral of its boundary values?

On reflection, it soon becomes clear that this is too ambitious since things can easily get out of control as we approach the boundary of D. For example, $F(z) = 1/(1 - z)$ is harmonic (even holomorphic) on D, but its boundary values $f(e^{ix}) = 1/(1 - e^{ix})$ fail to be integrable near $x = 0$ ($f(e^{ix}) \simeq i/x$ there), so we are not in the framework of our theory. Come to think of it, we actually do not obtain arbitrary harmonic functions as Poisson integrals $F = P_r * f$. Rather, we have at least some kind of average control when we approach the boundary in the sense that if $f \in L^p(T)$, then

$$\|f_r\|_p = \|P_r * f\|_p \le \|f\|_p, \qquad f_r(e^{ix}) = F(re^{ix}),$$

since $\|P_r\|_1 = 1$. In this setting, everything now works beautifully.

Theorem 2.2. *Let $1 < p \le \infty$. Suppose that $F : D \to \mathbb{C}$ is harmonic, and*

$$\sup_{0\le r<1} \int_T |F(re^{ix})|^p \, d\sigma(x) < \infty. \tag{2.7}$$

Then $f(e^{ix}) = \lim_{r\to 1-} F(re^{ix})$ *exists for almost all* x, $f \in L^p(T)$, *and* $F = P_r * f$.

Proof. This proof will depend on some abstract machinery (dual spaces and the Banach-Alaoglu theorem). Take a look at chapter 4 of my Functional analysis notes if you want to brush up on this. But the argument should also make some sense if you don't pay attention to the precise inner workings of these steps.

By (2.7), the collection of functions $f_r(e^{ix}) = F(re^{ix}) \in L^p(T)$, $0 \le r < 1$, is bounded in $L^p(T)$, that is, $\|f_r\|_p \le C$, with C independent of r. We have $L^p = (L^q)^*$, with $1/p + 1/q = 1$, that is, L^p is the dual space of L^q. Now the Banach-Alaoglu theorem (and the separability of the spaces involved, if you want to be really careful) implies that we can select a sequence $r_n \to 1-$ such that $f_{r_n} \to g$ in weak-$*$ sense, for some $g \in L^p$. More explicitly, this means that

$$\int_T h(e^{it}) f_{r_n}(e^{it})\, d\sigma(t) \to \int_T h(e^{it}) g(e^{it})\, d\sigma(x) \tag{2.8}$$

for every $h \in L^q$ (note also that the product of an L^q function with an L^p function is in L^1, by Hölder's inequality, so everything is well defined here).

Now let's look at $G(z) = P_r * g$. The (shifted) Poisson kernel $P_r(e^{i(x-t)})$, as a function of e^{it}, for fixed r, x, is an admissible h in (2.8). In fact, $P_r \in C(T)$, so certainly $P_r \in L^q$, as required. Thus, by (2.8),

$$G(z) = \lim_{n\to\infty} \int_T P_r(e^{i(x-t)}) f_{r_n}(e^{it})\, d\sigma(t).$$

On the right-hand side, we are looking at $P_r * f_{r_n}$. Now $f_{r_n} \in C(T)$, so, as we discussed, this Poisson integral solves the Dirichlet problem on D with boundary values f_{r_n}. But obviously the unique solution to this problem is $F(r_n z)$: this is still harmonic, by elementary calculus, and has the right boundary values on $|z| = 1$. Thus $G(z) = \lim F(r_n z) = F(z)$; recall that F, being harmonic, is continuous on D.

We have shown that $F = P_r * g$, for some $g \in L^p$. Now the rest follows from Theorem 2.1: the function g can be recovered as the boundary value of its Poisson integral, which is F, so $g(x) = \lim_{r\to 1-} F(re^{ix})$, with convergence both in L^p (if $p < \infty$) and pointwise almost everywhere. □

For $p = 1$, the argument does not work in literally this form because $L^1(T)$ is not a dual space (that is, there is no Banach space X such that X^* is isometrically isomorphic to $L^1(T)$). However, a slight adjustment in the set-up will salvage matters anyway. Namely, $L^1(T)$ is contained as a closed subspace in $M(T)$, the space of complex Borel measures on T, if we view an $f \in L^1(T)$ as the measure $f(e^{ix})\, d\sigma(x)$. Recall that a complex measure assigns a complex number $\mu(B)$ to a Borel set $B \subseteq T$. Its *total variation* $|\mu|$ is a finite positive measure on T, and $d\mu(x) = e^{i\alpha(x)}\, d|\mu|(x)$ for some function α. The norm of a $\mu \in M(T)$ is defined as $\|\mu\| = |\mu|(T)$.

Everything we discussed so far could have been done, more generally, on $M(T)$. The Fourier coefficients of a $\mu \in M(T)$ are defined as

$$a_n(\mu) = \int_T e^{-inx}\, d\mu(x).$$

We still have $\|a\|_\infty \le \|\mu\|$, as in Theorem 1.2, but the Fourier coefficients of a measure need not go to zero. For example, if $\mu = \delta_0$ is the point mass at $x = 0$, then $a_n = 1$.

Theorem 1.5 also holds for measures, with essentially the same proof as before, so a measure can be uniquely reconstructed (in principle, that is) from its Fourier coefficients. The Poisson integral of a $\mu \in M(T)$ is defined, as expected, as

$$(P_r * \mu)(e^{ix}) = \int_T P_r(e^{i(x-t)})\, d\mu(t). \tag{2.9}$$

This is still a harmonic function of $z = re^{ix}$, and it satisfies (2.7) for $p = 1$, as we can confirm by integrating (2.9) and changing the order of integration.

This extension of the theory to measures is what we need to make the argument from the proof of Theorem 2.2 work for $p = 1$ also because, unlike $L^1(T)$, the space $M(T)$ is a dual space. More precisely, $M(T) = C(T)^*$, and this (major) result is usually called the Riesz representation theorem.

Theorem 2.3. *Suppose that $F : D \to \mathbb{C}$ is harmonic, and*

$$\sup_{0 \le r < 1} \int_T |F(re^{ix})|\, d\sigma(x) < \infty. \tag{2.10}$$

Then there is a unique $\mu \in M(T)$ such that $F = P_r * \mu$. We can obtain this measure as the weak-$*$ limit $d\mu(x) = \lim_{r \to 1-} F(re^{ix})\, d\sigma(x)$. More explicitly,

$$\int_T g(e^{ix})\, d\mu(x) = \lim_{r \to 1-} \int_T g(e^{ix}) F(re^{ix})\, d\sigma(x)$$

for all $g \in C(T)$.

Sketch of proof. With the above preparations in place, we can now repeat the argument from the proof of Theorem 2.2. Consider the complex measures $d\mu_r(x) = F(re^{ix})\,d\sigma(x)$. By (2.10), this collection is bounded in $M(T)$, so will converge to a limit $\mu \in M(T)$, in weak-$*$ sense, along a suitable sequence $r_n \to 1-$. This means that

$$\int_T g(e^{it})F(r_n e^{it})\,d\sigma(t) \to \int_T g(e^{it})\,d\mu(t)$$

for all $g \in C(T)$. In particular, we can apply this to shifted Poisson kernels $g = P_r(e^{i(x-t)})$, and then we see as above that $F = P_r * \mu$. By the (undiscussed, here) measure analog of Theorem 2.1, we can then obtain $d\mu = \lim f_r\,d\sigma$ as the weak-$*$ limit of its Poisson integral (no subsequence needed), and this also shows that μ is unique. □

It can also be shown that $f(e^{ix}) := \lim_{r\to 1-} F(re^{ix})$ exists for almost every x in the situation of Theorem 2.3, but this pointwise limit will not, in general, recover the whole measure μ. Rather, it will only give us its absolutely continuous part $f\,d\sigma$. To recover the full measure, the limit must be taken in weak-$*$ sense.

Let's look at this in a concrete example. Let's again take $\mu = \delta_0$, the point mass at $x = 0$ (or, equivalently, at $e^{ix} = 1 \in T$). Write $F(z) = P_r * \mu = P_r(e^{ix})$; compare (2.9). Now $P_r(e^{ix}) \to 0$ as $r \to 1-$ for $x \neq 0$, as we can see for example from (2.2). Thus the pointwise limit of $F(re^{ix})$ is zero almost everywhere. We only recover the measure $\mu = \delta_0$ if we take the weak-$*$ limit ($P_r(e^{ix})$ will be small away from $x = 0$, but close to this point it will be large, in such a way that the integral stays equal to 1 for all $r < 1$).

3. H^p spaces: two definitions

We are now ready for our main topic.

Definition 3.1. Let $1 \le p \le \infty$. We say that $f \in H^p = H^p(T)$ if $f \in L^p(T)$ and $a_n(f) = 0$ for $n < 0$.

So $H^p \subseteq L^p$, and in fact H^p is a *closed* subspace of L^p. This follows because if $f_n \to f$ in L^p, then also $f_n \to f$ in $L^1 \supseteq L^p$, and this in turn implies that $a_j(f_n) \to a_j(f)$ for all $j \in \mathbb{Z}$, by the (trivial) inequality from Theorem 1.2.

Exercise 3.1. Show more explicitly that convergence in L^p implies convergence in L^1.

This definition of the H^p spaces or *Hardy spaces* is the most convenient one for us, but it is not the one usually given. Rather, one defines

$H^p = H^p(D)$ as the space of holomorphic functions $F : D \to \mathbb{C}$ that satisfy

$$\sup_{0\le r<1} \|f_r\|_p < \infty, \qquad f_r(e^{ix}) = F(re^{ix}). \tag{3.1}$$

This latter condition looks familiar, of course; compare (2.7), (2.10).

The two definitions are equivalent, but clearly this claim needs some elaboration since the two versions don't even deal with the same kind of object: in Definition 3.1, our functions are defined on T, while the second version is about functions on D. However, the material of the previous section indicates how to go back and forth between these. The easier direction is the one where we start out with an $f \in H^p(T)$. We then form its Poisson integral

$$F(z) = (P_r * f)(e^{ix}), \quad z = re^{ix} \in D.$$

As we discussed in Section 2, we have the alternative formula $F(z) = \sum a_n(f) r^{|n|} e^{inx}$. We are currently assuming that $a_n(f) = 0$ for $n < 0$, so the sum is only over $n \ge 0$, and thus

$$F(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D.$$

This power series converges on D, as we knew from the beginning, but can also confirm one more time directly since a_n is bounded, so the radius of convergence is at least 1. So F is holomorphic on D, and thus $F \in H^p(D)$ since (3.1) holds for any Poisson integral $F = P_r * f$ with $f \in L^p$, as we discussed in Section 2. So, to summarize this step, any $f \in H^p(T)$ is naturally associated with an $F \in H^p(D)$, given by $F = P_r * f$.

Conversely, assume now that an $F \in H^p(D)$ is given. We want to take the above steps in reverse, so come up with an $f \in H^p(T)$ such that $F = P_r * f$. A holomorphic function is harmonic (but not conversely, in general), so Theorem 2.2 tells us that this representation will work for $1 < p \le \infty$, if we take $f(e^{ix}) = \lim_{r\to 1-} F(re^{ix})$; this limit will exist for almost every x. We also have uniqueness in the sense that only this f will give us $F = P_r * f$ as a Poisson integral. Moreover, $f \in L^p(T)$. Of course, we need more than that here: we want $f \in H^p(T)$, that is, we must show that, in addition, $a_n(f) = 0$ for $n < 0$. We can argue as above, when we showed that H^p is closed, and use the convergence in L^1 to conclude that

$$a_n(f) = \lim_{r\to 1-} a_n(P_r * f).$$

However, the Fourier coefficients of this *holomorphic* function $F = P_r * f$ are zero for $n < 0$ because they can be interpreted as a complex line integral

$$a_n = \int_T F(re^{ix})e^{-inx}\, d\sigma(x) = \frac{1}{2\pi i}\int_T F(rz)z^{-n-1}\, dz,$$

which equals zero, by Cauchy's theorem, since the integrand is holomorphic (recall that $-n-1 \geq 0$).

Exercise 3.2. Give an alternative proof that $a_n(f) = 0$ for $n < 0$, as follows: recall that

$$F(z) = P_r * f = \sum_{n\geq 0} a_n z^n + \sum_{n\leq -1} a_n \overline{z}^{|n|},$$

and then show that this expression is holomorphic only if $a_n = 0$ for $n < 0$.

So for $p > 1$, we have set up the promised one-to-one correspondence between $H^p(T)$ and $H^p(D)$. We pass from one realization of this space to the other with the help of the operations *take the Poisson integral* and *take the boundary values.* However, there seems to be a problem for $p = 1$. Everything works smoothly if $f \in H^1(T)$ is given, as we discussed: we obtain the associated $F \in H^1(D)$ as $F = P_r * f$. However, if we start out with an $F \in H^1(D)$ and refer to Theorem 2.3 to write this as a Poisson integral, then this result does not guarantee that $F = P_r * f$, with $f(e^{ix}) = \lim_{r\to 1-} F(re^{ix})$ being the boundary value almost everywhere of F (which we do know exists). Rather, $F = P_r * \mu$ for a *measure* μ, and $f\, d\sigma$ is only its absolutely continuous part.

While this is indeed the situation for *harmonic* functions $F : D \to \mathbb{C}$, for *holomorphic* functions $F \in H^p(D)$ this discrepancy between $p = 1$ and $p > 1$ goes away, and $p = 1$ is no longer special. This beautiful result is our first major theorem on H^p spaces.

Theorem 3.2 (F. and M. Riesz)**.** *Suppose that* $F \in H^1(D)$. *Then* $f(e^{ix}) = \lim_{r\to 1-} F(re^{ix})$ *satisfies* $f \in H^1(T)$ *and* $F = P_r * f$.

This can be rephrased as a statement about measures, which is equally satisfying.

Theorem 3.3. *Suppose that* $\mu \in M(T)$, $a_n(\mu) = 0$ *for all* $n < 0$. *Then* $d\mu = f\, d\sigma$, *for some* $f \in H^1(T)$.

Even the less precise statement that $d\mu = f\, d\sigma$ for some $f \in L^1(T)$, that is, μ is an absolutely continuous measure, is very interesting. This is a smoothness property of sorts, and one would normally think that this must have something to do with the decay of the Fourier coefficients as $n \to \pm\infty$, but the theorem obtains this conclusion from information on just one half of the Fourier coefficients.

Exercise 3.3. Explain in more detail how Theorems 3.2 and 3.3 imply each other. (That should also become clearer later on, once we discuss the proof of Theorem 3.3.)

With this correspondence between $H^p(D)$ and $H^p(T)$ now set up in all cases, we will usually not distinguish between the two versions and also often simply write H^p for this space, with the understanding that its elements can be thought of either as functions on T or as (holomorphic) functions on D, and we will indeed switch back and forth between the two interpretations quite routinely. However, we will continue to write $H^p(D)$ or $H^p(T)$ if the distinction matters and we want to emphasize it.

The spaces $H^p(D)$ are particularly easy to describe for $p = 2, \infty$: $H^\infty(D)$ is simply the space of bounded holomorphic functions $F : D \to \mathbb{C}$, and $H^2(D)$ is the space of all functions

$$F(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{3.2}$$

with $a_n \in \ell^2$. (Of course, the power series representation (3.2) is valid for $F \in H^p(D)$ for any p, but if $p \neq 2$, then we don't have a good description of the coefficient sequences a_n that occur.)

We will return to Theorem 3.3 later. For now, we'll look at a rather different topic.

4. Invariant subspaces

If $A : H \to H$ is a bounded linear operator on a Hilbert space H, then a closed subspace $M \subseteq H$ is called an *invariant subspace* if $AM \subseteq M$, that is, $Ax \in M$ for all $x \in M$. For example, if A is a matrix, acting on $H = \mathbb{C}^n$, then any subspace spanned by some (but not necessarily all) of the eigenvectors of A is an invariant subspace. In particular, if $H = \mathbb{C}^n$ is finite-dimensional, then non-trivial invariant subspaces always exist. Here, we call M non-trivial if $M \neq 0, H$. For infinite-dimensional separable Hilbert spaces such as $H = L^2(T)$, $H = \ell^2$ the corresponding question *does every bounded operator have a non-trivial invariant subspace?* has been a long-standing and famous open problem.

There is one easy general procedure to produce invariant subspaces. Namely, start out with an arbitrary $f \in H$, keep applying the operator, that is, form $f, Af, A^2f, \ldots$, and then take the closed linear span of this sequence. We'll denote this subspace by $M_f \subseteq H$ (it obviously also depends on A).

Exercise 4.1. Prove more explicitly that M_f is an invariant subspace. (This is purely a Functional analysis exercise.)

Of course, this does not solve the invariant subspace problem because it's completely possible that $M_f = H$.

For $H = H^2(T)$ or $H = L^2(T)$, the operator A of multiplication by e^{ix}, $(Af)(e^{ix}) = e^{ix}f(e^{ix})$ is obviously bounded; in fact $|(Af)(e^{ix})| = |f(e^{ix})|$, so $\|Af\|_2 = \|f\|_2$, that is, A is an *isometry* (preserves norms).

Exercise 4.2. Show that if the realization $H^2(D)$ of H^2 is used instead, then (unsurprisingly) $(Af)(z) = zf(z)$.

The invariant subspaces of A on H^2 can be determined; this is a famous result of Beurling. We can also find the additional invariant subspaces of A on the larger space L^2, which is an easier result (certainly from an abstract, spectral theoretic point of view, since A is unitary on L^2, but not on H^2). We'll combine the two parts here:

Theorem 4.1. *Each invariant subspace M of $A : L^2(T) \to L^2(T)$ is of one of two types. Either $M = L^2(B)$ for some (Borel) subset $B \subseteq T$, or $M = qH^2(T)$, for some $q \in L^\infty(T)$ with $|q(e^{ix})| = 1$ almost everywhere on T.*

Here we of course think of $L^2(B) \subseteq L^2(T)$ as a subspace of $L^2(T)$, by identifying an $f \in L^2(B)$ with $f_0 \in L^2(T)$, $f_0 = f$ on B, $f_0 = 0$ on $T \setminus B$.

Exercise 4.3. Prove that any space of the type $M = L^2(B)$ or $M = qH^2$ is invariant under A (this easy fact is of course more or less assumed in the formulation of the theorem).

The second type of space, qH^2 is the same as $M_q = \overline{L(q, zq, z^2q, \ldots)}$, as defined above. To prove this remark, observe first that $qH^2 \subseteq L^2$ is a *closed* subspace: if $f_n \in H^2$, $qf_n \to g \in L^2$, then also $f_n \to g/q$ in L^2, since $|q| = 1$. But $H^2 \subseteq L^2$ is closed, so $g/q \in H^2$ and thus $g \in qH^2$, as claimed.

Now $1, z, z^2, \ldots \in H^2$, and thus $M_q \subseteq qH^2$ (and for this step, I use that qH^2 is closed). Conversely, any $f(z) = \sum_{n=0}^\infty a_n z^n \in H^2$ is the norm limit $f = \lim S_N$, $S_N = \sum_{n=0}^N a_n z^n$, of finite linear combinations of the z^n, $n \geq 0$, and then also $qS_N \to qf$ in L^2, by the same argument as above, since $|q| = 1$. It follows that $qH^2 \subseteq M_q$.

We have

$$qH^2 \subseteq H^2 \quad \text{if and only if} \quad q \in H^2, \tag{4.1}$$

if q is as in Theorem 4.1, that is, $|q| = 1$ on T. Clearly, the condition $q \in H^2$ is necessary since $1 \in H^2$, so $q \in qH^2$. The converse follows from the following general fact, which we'll use repeatedly in the sequel.

Lemma 4.2. *Suppose that $f, g \in H^2$. Then $fg \in H^1$.*

This formulation almost slightly obscures what the lemma is about. Recall that an $f \in L^p$ is in $H^p(T)$ if and only if its negative Fourier coefficients are all zero. The lemma really says that the product of two such functions with no negative Fourier coefficients has the same property.

Exercise 4.4. Prove the following precise version of this remark, assuming Lemma 4.2: if $f, g \in H^2(T)$ and $fg \in L^p(T)$ for some $1 \le p \le \infty$, then $fg \in H^p(T)$.

This can look completely trivial if we are just a bit superficial: multiplying $\sum_{n\ge 0} a_n e^{inx}$ and $\sum_{n\ge 0} b_n e^{inx}$ (formally) produces another such series with no $n < 0$ terms. But we mustn't forget that these Fourier series are not guaranteed to converge (at least, not for L^1 functions), so a more solid argument should be given.

Proof of Lemma 4.2. First of all, observe that if $f_n \to f$, $g_n \to g$ in L^2, then $f_n g_n \to fg$ in L^1. This follows from the Cauchy-Schwarz inequality (which will also make sure that these products are in L^1):

$$\begin{aligned} \int |f_n g_n - fg| &\le \int |f_n - f|\,|g_n| + \int |f|\,|g_n - g| \\ &\le \|f_n - f\|_2 \|g_n\|_2 + \|f\|_2 \|g_n - g\|_2 \to 0 \end{aligned}$$

As we observed earlier, $f, g \in H^2$ can be approximated in norm by (trigonometric) polynomials $P_k \to f$, $Q_k \to g$. In fact, all we need to do is cut off the Fourier (or Taylor) series $f = \sum_{n\ge 0} a_n z^n$. Now clearly the product of two such *finite* sums is another such (finite) sum with no $n < 0$ terms, so $a_n(P_k Q_k) = 0$ for $n < 0$. These Fourier coefficients converge to $a_n(fg)$ as $k \to \infty$, by our earlier observation that $P_k Q_k \to fg$ in L^1 and Theorem 1.2. □

Now let's return to our discussion of (4.1). If $q \in H^2$, $|q| = 1$, and $f \in H^2$, then clearly $qf \in L^2$, and now Lemma 4.2 in the version of Exercise 4.4 shows that $qf \in H^2$.

Such a function $q \in H^2$, $|q| = 1$, actually lies in H^∞, by the argument from Exercise 4.4 again: $a_n(q) = 0$ for $n < 0$ since $q \in H^2$, and obviously $q \in L^\infty(T)$, hence $q \in H^\infty(T)$.

Recall that the *maximum principle* for holomorphic functions says that if f is holomorphic on a neighborhood of a compact set K, then the maximum of $|f|$ on K can only be assumed on the boundary of K, unless f is constant.

Exercise 4.5. Deduce from the maximum principle that if $F \in H^\infty(D)$ with boundary values f, then

$$\sup_{z\in D} |F(z)| = \|f\|_{L^\infty(T)};$$

please pay attention to the small details here: the right-hand side is the *essential supremum* of the boundary function $f(e^{ix}) = \lim_{r\to 1-} F(re^{ix})$, and we know that this limit exists almost everywhere.

Theorem 4.1 does not bring us any closer to a resolution of the invariant subspace problem, but this context explains some of the excitement surrounding it. Also, results like this one have inspired the hope that complex analytic methods may be relevant to the invariant subspace problem (though the proof we are going to discuss will actually use hardly any complex analysis at all).

Exercise 4.6. Consider the unitary map $U : L^2(T) \to \ell^2(\mathbb{Z})$ that sends an $f \in L^2$ to its Fourier coefficients, so $(Uf)_n = a_n(f)$ (see also Theorem 1.1). Show that $S = UAU^*$ (and $U^* = U^{-1}$ for a unitary operator), that is, the operator A realized on $\ell^2(\mathbb{Z})$, is the shift operator $(Sa)_n = a_{n-1}$. So Theorem 4.1 also determines the invariant subspaces of the shift operator. Also, what is UH^2?

Proof of Theorem 4.1. Let $M \subseteq L^2(T)$ be an invariant subspace, so $e^{ix}M \subseteq M$. Let's first consider the case when also $e^{-ix}M \subseteq M$. By repeatedly applying these properties, we see that then also $e^{inx}M \subseteq M$ for any $n \in \mathbb{Z}$.

Now take any $f \in M$, $g \in M^\perp$ (that is, $\langle g, m\rangle = 0$ for all $m \in M$). Since then also $fe^{inx} \in M$, as we just observed, we have

$$0 = \langle g, fe^{inx}\rangle = \int_T \overline{g} f e^{inx}\, d\sigma.$$

In other words, all Fourier coefficients of the function $\overline{g}f \in L^1(T)$ are equal to zero. Hence $\overline{g}f = 0$, by Theorem 1.5.

Consider now (Borel) sets $B \subseteq T$ with the property that there is an $f \in M$ with $f \neq 0$ almost everywhere on B. Let's denote the collection of these sets by $\mathcal{A}$, and define $m = \sup_{B\in\mathcal{A}} \sigma(B)$. Pick a sequence $B_n \in \mathcal{A}$ with $\sigma(B_n) \to m$, and let $B = \bigcup_{n\geq 1} B_n$. Then $\sigma(B) \geq m$ (prove this in detail perhaps as an exercise in basic measure theory). If now $g \in M^\perp$ is arbitrary, then $g = 0$ almost everywhere on B_n, by what we showed in the previous paragraph (since there is an $f_n \in M$ with $f_n \neq 0$ almost everywhere on B_n, but we must also have $\overline{g}f_n = 0$). Thus in fact $g = 0$ almost everywhere on B, and this means that $\langle g, f\rangle = 0$ for any $f \in L^2(B)$. Since $g \in M^\perp$ was arbitrary, this says that any $f \in L^2(B)$ belongs to $M^{\perp\perp} = M$. This last equality follows because M is a closed subspace. So $L^2(B) \subseteq M$.

To establish the reverse inclusion, let $f \in M$ be arbitrary. Note that then also $f + g \in M$ for any $g \in L^2(B)$. So if we had $f \neq 0$ on a positive measure subset of $T \setminus B$, then we would also obtain a function $h \in M$ with $h \neq 0$ on a set $C \supsetneq B$, and $\sigma(C) > \sigma(B) \geq m$ (just take something like $g = \chi_N$ above, where $N = \{e^{ix} \in B : f(e^{ix}) = 0\}$). This contradicts the definition of m. It follows that $f = 0$ on B^c for any $f \in M$, and thus $M = L^2(B)$.

Now let's move on to the other case, when $e^{-ix}M \not\subseteq M$. Then also $e^{ix}M \subsetneq M$, and since both sets are closed subspaces, we can find a $q \in M$, $q \neq 0$, $q \perp e^{ix}M$. We can also demand here that $\|q\|_2 = 1$. The subspace M is invariant and $q \in M$, so $qe^{inx} \in M$ for $n \geq 0$ and $qe^{inx} \in e^{ix}M$ for $n \geq 1$. Thus, for $n \geq 1$, we have

$$0 = \langle q, qe^{inx}\rangle = \int_T |q|^2 e^{inx}\, d\sigma.$$

By taking complex conjugates, we obtain this condition also for $n \leq -1$. In other words, the only Fourier coefficient of the function $|q|^2 \in L^1(T)$ that can be non-zero is $a_0(|q|^2)$, and this means, by uniqueness, that $|q|^2$ is a constant function. Since $\|q\| = 1$, we have $|q| = 1$.

We know that the exponentials e^{inx}, $n \in \mathbb{Z}$, form an ONB of $L^2(T)$; compare Theorem 1.1. This implies that $\{q(e^{ix})e^{inx} : n \in \mathbb{Z}\}$ is another ONB: these functions are still orthonormal because when we work out their scalar products, we only pick up an additional factor $|q|^2 = 1$, and their closed linear span is still all of $L^2(T)$ because if $f \in L^2(T)$ is given, then also $f/q \in L^2(T)$, and then we expand $f/q = \sum a_n e^{inx}$ into exponentials to obtain an expansion of f in terms of the functions qe^{inx} (and this second part of the argument also depends on the fact that $|q| = 1$).

Now $qe^{inx} \in M$ for $n \geq 0$, and if $n < 0$, then $qe^{inx} \in M^\perp$. To check this second claim, take an arbitrary $f \in M$ and compute

$$\langle qe^{inx}, f\rangle = \int_T \overline{q}e^{-inx} f\, d\sigma = \langle q, e^{-inx}f\rangle = 0,$$

since $-n \geq 1$, so $e^{-inx}f \in e^{ix}M$, and we took $q \perp e^{ix}M$.

It follows that $M = \overline{L(qe^{inx} : n \geq 0)}$.

Exercise 4.7. Provide more details for this step.

So, if we use the notation introduced above, then $M = M_q$, with $|q| = 1$ on T, and we established earlier that this space equals qH^2. $\square$

We have also seen that these two types of invariant subspaces don't overlap, that is, if $M = L^2(B)$, then there is no q, $|q| = 1$, such that also $M = qH^2$, and vice versa. Indeed, if $f \in L^2(B)$, then also $e^{-inx} f \in L^2(B)$ for arbitrarily large n, but this will not hold for an $f \in qH^2$, $f \neq 0$: in that case $f = qg$, $g \in H^2$, so g has non-zero Fourier coefficients only for non-negative integers, but of course multiplying by e^{-inx} will eventually produce Fourier coefficients at negative integers if $g \neq 0$. Or, to state this more succinctly, an invariant subspace either is invariant under multiplication by e^{-ix} also, or it isn't, but not both.

Theorem 4.3. *Let $f \in H^2(T)$. If $f = 0$ on a set of positive measure, then $f \equiv 0$.*

This is also valid for $f \in H^1$, as we will prove later. The statement can be viewed as a version of the identity theorem for holomorphic functions, which says that if $F : D \to \mathbb{C}$ is holomorphic and $F = 0$ on a set with an accumulation point in D, then $F \equiv 0$. Our functions are not holomorphic on T, but they are boundary values of holomorphic (on D) functions, and we still have a statement saying that the function cannot be zero very frequently without being zero identically.

Proof. Let $f \in H^2(T)$, $f \not\equiv 0$, and consider $M_f = \overline{L(f, zf, z^2 f, \ldots)}$. As we discussed earlier, M_f is an invariant subspace, and $M_f \subseteq H^2$. The arguments we just discussed, in the paragraph preceding the theorem, also show that such a subspace cannot be of the type $L^2(B)$. Thus $M_f = qH^2$ for some q with $|q| = 1$, by Theorem 4.1. In particular, $q \in M_f$, and this function does not vanish anywhere. If we had $f = 0$ on some set $B \subseteq T$, $\sigma(B) > 0$, then the same would be true for all $g \in M_f$ (for example because a norm convergent sequence also converges almost everywhere along a subsequence). So this is not possible here. $\square$

5. Factorization and the F. and M. Riesz theorem

Definition 5.1. We call $f \in H^2(T)$ an *inner function* if $|f| = 1$ almost everywhere and an *outer function* if $M_f = H^2$.

Note that the functions q from Theorem 4.1 are not necessarily inner functions because Definition 5.1 also insists that $q \in H^2$. Recall also that for any $f \in H^2$, outer or not, we have $M_f \subseteq H^2$.

Theorem 5.2. *Let $f \in H^2(T)$, $f \not\equiv 0$. Then we can factor $f = qg$, with q inner and g outer. This factorization is essentially unique in the sense that if also $f = ph$ with p inner, h outer, then $p = e^{i\alpha} q$, $h = e^{-i\alpha} g$ for some $\alpha \in \mathbb{R}$.*

Proof. Consider the invariant subspace M_f. Since $M_f \subseteq H^2$, this must be of the type $M_f = qH^2$ for some q with $|q| = 1$. In fact, $q = q \cdot 1 \in M_f \subseteq H^2$, so q is inner. Since $f \in M_f = qH^2$, we can write $f = qg$ for some $g \in H^2$. We want to show that g is outer, so let's look at M_g. We have

$$M_g = M_{f/q} = q^{-1}M_f = q^{-1}(qH^2) = H^2,$$

as desired.

Exercise 5.1. Explain the second equality in more detail. Use the definition of M_h and the fact that $|q| = 1$.

If also $f = ph$, with p inner, h outer, then $qg = ph$, so $(q/p)g = h$ is outer, and we see in the same way that

$$H^2 = M_{(q/p)g} = (q/p)M_g = (q/p)H^2.$$

Hence $q/p = (q/p) \cdot 1 \in H^2$. In the same way, we can show that $p/q \in H^2$. Now $|p| = |q| = 1$, so $1/p = \overline{p}$ and $1/q = \overline{q}$, so we can slightly rephrase and say that $p\overline{q}, \overline{p}q \in H^2(T)$. These two functions will also be complex conjugates of one another if considered as functions on D, so if we pass to this picture, then we are dealing with an $S = P\overline{Q} \in H^2(D)$ with the property that $\overline{S} \in H^2(D)$ also. This follows because we pass to S by taking the Poisson integral $S = P_r * s$, and since the Poisson kernel is real, we have $P_r * \overline{s} = \overline{P_r * s}$.

However, it is not possible for a function and its complex conjugate to be holomorphic simultaneously unless the function is constant. Thus $p/q = c$, and of course this constant $c = e^{i\alpha}$ must be of absolute value 1 here. □

Exercise 5.2. Brush up your complex analysis if needed and give a proof of this fact that we just used: if both $F : D \to \mathbb{C}$ and $\overline{F}$ are holomorphic, then $F \equiv c$. *Suggestion:* Consider $F \pm \overline{F}$ and apply the open mapping theorem.

Exercise 5.3. In the situation of Theorem 5.2, let $F, Q, G \in H^2(D)$ be the associated functions on D. Show that then also $F(z) = Q(z)G(z)$ for all $z \in D$.

Such a factorization also works on H^1. Of course, something must be modified here since the products qg, q inner, g outer, give us exactly H^2, as we just proved, and $H^1 \supsetneq H^2$. The definition of an inner function doesn't seem to be tied to specifically H^2 very closely, and in fact $q \in H^\infty$ if q is inner, so if we want to tamper with Definition 5.1, we'd probably have to modify the definition of outer functions. That can be done, but an easier workaround is more convenient for us here.

Theorem 5.3. *Let $f \in H^1(T)$, $f \not\equiv 0$. Then we can factor $f = qg^2$, with q inner, g outer.*

At least in hindsight, the appearance of the extra square is quite plausible. H^1 functions are in $L^1(T)$, and the L^1 functions are exactly the squares of the L^2 functions.

As an immediate payoff, we obtain the promised more general version of Theorem 4.3 from this.

Corollary 5.4. *Let $f \in H^1(T)$, $f \not\equiv 0$. Then $f \neq 0$ almost everywhere on T.*

Proof. Write $f = qg^2$, as in Theorem 5.3. Since $|q| = 1$, if we had $f = 0$ on a positive measure set, then also $g = 0$ on this set, so $g \equiv 0$ by Theorem 4.3 and thus also $f \equiv 0$. □

Proof of Theorem 5.3. Let $f \in H^1$, $f \not\equiv 0$. Define $w = |f|^{1/2} \in L^2(T)$, and write $f = p|f| = pw^2$, with $|p| = 1$ on T. Since $f \in H^1$, we have

$$0 = a_{-n}(f) = \int_T f e^{inx}\, d\sigma = \int_T (pw)(we^{inx})\, d\sigma$$

for all $n \geq 1$. In other words, $\overline{pw} \perp we^{inx}$. These functions we^{inx}, $n \geq 1$, span the invariant subspace $M_{we^{ix}}$, so we have shown that $\overline{p}w$ lies in the orthogonal complement of this space. This rules out the scenario where $M_{we^{ix}} = L^2(B)$ because then $\overline{p}w$ would have to be zero (almost everywhere) on B while $we^{ix} = 0$ on B^c, so $w \equiv 0$ and then also $f \equiv 0$, contrary to our assumption.

So we conclude that $M_{we^{ix}} = q_0 H^2$. In particular, $we^{ix} = q_0 g$ for some $g \in H^2$, or we can say that $w = qg$, with $q = q_0 e^{-ix}$, $|q| = 1$. In fact,

$$M_g = M_{w/q} = q^{-1} M_w = q^{-1} e^{-ix} M_{we^{ix}} = (q_0/q) e^{-ix} H^2 = H^2,$$

so g is outer. We now have the presentation $f = pw^2 = pq^2 g^2$, so we can finish the proof by showing that pq^2 is inner. We of course already know that $|pq^2| = 1$, so we must show that $pq^2 \in H^2$.

The condition that g is outer means, more explicitly, that any $h \in H^2$ can be approximated, in H^2, by functions of the form $g \sum_{n=0}^{N} a_n e^{inx}$. Since $1 \in H^2$, we can, in particular, find such finite sums P_n (P as in *polynomial,* because this is what it is if thought of as a function of $z \in D$) such that $\|gP_n - 1\|_2 \to 0$. Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \|fP_n - pq^2 g\|_1 &= \|pq^2 g(gP_n - 1)\|_1 \leq \|pq^2 g\|_2 \|gP_n - 1\|_2 \\ &= \|g\|_2 \|gP_n - 1\|_2 \to 0. \end{aligned}$$

Moreover, $fP_n \in H^1$ because this function is a finite linear combination of functions of the form fe^{inx}, with $n \ge 0$, which are in $H^1(T)$ (multiplying by e^{inx}, $n \ge 0$, shifts the Fourier coefficients to the right, so does not produce Fourier coefficients for negative n). Since H^1 is closed in L^1, it follows that $pq^2 g \in H^1$. But we also know that $pq^2 g \in L^2$, so in fact $pq^2 g \in H^2$. Thus also $pq^2 g P_n \in H^2$ for each n, by the same argument that we just gave for fP_n. Finally,

$$\|pq^2 g P_n - pq^2\|_2 = \|pq^2(gP_n - 1)\|_2 = \|gP_n - 1\|_2 \to 0,$$

so $pq^2 \in H^2(T)$ as well, as we wished to show. □

Theorem 5.5. *Let $F \in H^2(D)$ be an outer function. Then $F(z) \neq 0$ for all $z \in D$.*

Proof. The map $H^2 \to \mathbb{C}$, $F \mapsto F(z)$, is of the form $F(z) = (P_r * f)(e^{ix}) = \langle P_r(e^{i(x-t)}), f\rangle$ and thus is a continuous linear functional on H^2 for any fixed $z \in D$.

Any $G \in M_f$ is a norm limit of linear combinations of functions $z^n F(z)$, $n \ge 0$, so if we had $F(z_0) = 0$ for some $z_0 \in D$, then $G(z_0) = 0$ for all $G \in M_f$. This is clearly impossible when F is outer because then $M_f = H^2$, and H^2 contains functions that do not vanish at z_0 (such as $G = 1$). □

We are now finally ready for the proof of the theorem of F. and M. Riesz. We'll prove it in the version of Theorem 3.3. If we have an $F \in H^1(D)$, as in Theorem 3.2, then $F = P_r * \mu = \sum_{n\in\mathbb{Z}} a_n(\mu) r^{|n|} e^{inx}$, by Theorem 2.3, and the assumption that F is a holomorphic function of $z = re^{ix}$ will now imply that $a_n(\mu) = 0$ for all $n < 0$. Then Theorem 3.3 will show that $F = P_r * f$ for some $f \in H^1(T)$, but then we also know that we can obtain f as the boundary value of its Poisson integral, and Theorem 3.2 follows.

Proof of Theorem 3.3. Consider the holomorphic function

$$F(z) = (P_r * \mu)(e^{ix}) = \sum_{n=0}^{\infty} a_n(\mu) z^n, \qquad z = re^{ix} \in D. \tag{5.1}$$

Since $\int_T |P_r|\, d\sigma = 1$, we have

$$\int_T |F(re^{ix})|\, d\sigma(x) \le |\mu|(T) = \|\mu\|, \tag{5.2}$$

so $F \in H^1(D)$. As a first preparatory step, I want to establish a factorization similar to Theorem 5.3 for such an F. More precisely, we will prove that

$$F(z) = Q(z)G^2(z), \tag{5.3}$$

with $Q, G \in H^2(D)$, $|Q| \le 1$. This we will do by a limiting procedure. For fixed $r < 1$, we have $F(rz) \in H^2(D)$ (in fact, we have more, the function is continuous, but this will be enough for our purposes). So $f_r = q_r h_r$, with $q_r \in H^2(T)$ inner, $h_r \in H^2(T)$ outer, and here I write $f_r(z) = F(rz)$, as usual, for the boundary values of the function $F(rz)$. The corresponding function $H_r \in H^2(D)$ has no zeros on D, by Theorem 5.5, and thus we may take a holomorphic square root. In other words, we can write $H_r = G_r^2$, with $G_r : D \to \mathbb{C}$ holomorphic. Here $\|g_r\|_2^2 = \|h_r\|_1 = \|f_r\|_1 \le C$, by (5.2).

Now we use one more time the functional analytic machinery from the proof of Theorem 2.2. On a suitable sequence, g_{r_n} will converge weakly in $L^2(T)$ to a limit $g \in L^2(T)$. More explicitly, this means that

$$\int_T k g_{r_n}\, d\sigma \to \int_T kg\, d\sigma \tag{5.4}$$

for all $k \in L^2(T)$. Since a closed subspace is also weakly closed, we in fact have $g \in H^2(T)$.

Exercise 5.4. Give a proof of this fact if you want to. So let $M \subseteq H$ be a closed subspace of a Hilbert space H, let $x_n \in M$ and suppose that $\langle y, x_n\rangle \to \langle y, x\rangle$ for all $y \in H$, for some $x \in H$. Show that then $x \in M$. *Suggestion:* What can you say about $\langle y, x\rangle$ for $y \in M^\perp$?

Since the Poisson kernel is a possible choice of k in (5.4), we also have pointwise convergence $G_{r_n}(z) \to G(z)$ at each $z \in D$ for the corresponding functions $G_{r_n}, G \in H^2(D)$.

Now look at what happens if we send $r \to 1-$ along the sequence r_n in $F(rz) = Q_r(z)G_r(z)^2$. Obviously, $F(rz) \to F(z)$. This shows, first of all, that $G \equiv 0$ is not possible, and then we deduce that $Q_{r_n}(z)$ converges also, at least away from the zeros of G, to $Q := F/G^2$. This function is meromorphic, but we also know that $|Q_r| \le 1$, so $|Q| \le 1$, and thus Q is in fact holomorphic, or, to say the same thing slightly differently, $Q \in H^\infty(D)$. We have now proved (5.3).

Let's now focus on the boundary values f, q, g. We have $qg \in H^2(T)$, by Lemma 4.2 and Exercise 4.4, and this in turn implies that $f = qg \cdot g \in H^1(T)$, by the same result. So, to summarize, we have now shown that the F from (5.4) (which is the general function $F \in H^1(D)$, by our discussion preceding the proof) has boundary values $f \in H^1(T)$.

These boundary values $f(e^{ix}) = \lim_{r\to 1-} F(re^{ix})$ are approached not just pointwise, but also in $L^1(T)$. This follows because $G \in H^2(D)$ is the Poisson integral $G = P_r * g$ of its boundary values $g \in H^2(T)$, so $G(re^{ix}) \to g$ in $L^2(T)$ by Theorem 1.3. Moreover, $|Q| \le 1$, so, writing

$f_r(e^{ix}) = F(re^{ix})$, as usual, we indeed have

$$\|f_r - f\|_1 \le \|g_r^2 - g^2\|_1 = \|(g_r + g)(g_r - g)\|_1 \le (\|g_r\|_2 + \|g\|_2) \|g_r - g\|_2 \to 0.$$

Now $f_s \in C(T)$ is well behaved for any $s < 1$, so $F(sz) = P_r * f_s$, $z = re^{ix}$, certainly is the Poisson integral of its boundary values, and we can now send $s \to 1-$ to obtain the desired conclusion that $F(z) = P_r * f$, with $f \in H^1(T)$ being the boundary value of F.

Exercise 5.5. Give a more detailed argument for the fact I just used, namely that $(P_r * h_n)(e^{ix}) \to (P_r * h)(e^{ix})$ if $h_n \to h$ in $L^1(T)$. This is easy, you just need to make sure that you won't get confused by the notation. In particular, note that r, e^{ix} are held fixed here.

Or, to state this in exactly the same form as in Theorem 3.3, we can say that $d\mu = f\, d\sigma$ because also $F = P_r * \mu$ and the measure in a Poisson representation is unique. □

6. Outer functions

Let $F \in H^\infty(D) \subseteq H^2(D)$ be a bounded outer function. Let's in fact assume that $|F| \le 1$, which we can of course always achieve by multiplying F by a constant. F has no zeros on D, by Theorem 5.5, so we can take a holomorphic logarithm $\log F(z)$. This function is not uniquely determined: we can add multiples of $2\pi i$. We simply fix one such choice.

For each $r < 1$, we then have

$$\log |F(0)| = \int_T \log |F(re^{ix})|\, d\sigma(x). \tag{6.1}$$

Since $P_0 = 1$, this is the Poisson representation formula, at $z = 0$, for the bounded harmonic function $\log |F(rz)| = \text{Re } \log F(rz)$. Alternatively, we can view (6.1) as the mean value property for $\log |F(rz)|$.

Now $\log |F| \le 0$ since $|F| \le 1$, so we can apply Fatou's lemma to the functions $-\log |F(re^{ix})|$, which converge pointwise almost everywhere to $-\log |f(e^{ix})|$, with $f \in H^\infty(T)$ denoting the boundary values of F, as usual. This gives

$$-\int_T \log |f(e^{ix})|\, d\sigma(x) \le -\log |F(0)|;$$

in particular, $\log |f| \in L^1(T)$.

We also see from (6.1) that the harmonic function $\log |F(z)|$ satisfies (2.10), which is the assumption of Theorem 2.3 (with, of course, $\log |F|$ taking the role of what we called F there). By that result, $\log |F| =$

$P_r * \mu$ for some measure μ, which we can obtain as the weak-$*$ limit of the measures $\log|f_r|\,d\sigma$, with $f_r(z) = F(rz)$, as usual. We now want to show that in fact $d\mu = \log|f|\,d\sigma$ here.

To do this, let's start out at the other end, and let's look at $P_r * \log|f|$. In fact, let's do this in more abstract fashion. Let $w \in L^\infty(T)$, $0 < w \leq 1$, $\log w \in L^1(T)$ (we just showed that $w = |f|$ has these properties, if $|F| \leq 1$ is outer). Then let's define $G = P_r * \log w$. This is a real valued (in fact, negative) harmonic function, and I want to provide a *harmonic conjugate,* that is, come up with another harmonic function $\widetilde{G}$ such that $G + i\widetilde{G}$ is holomorphic. It is a basic result of complex analysis that such harmonic conjugates exist and are unique, up to an additive constant, on simply connected domains such as D (but they won't exist in general, and for a counterexample, you can consider the harmonic function $F : D \setminus \{0\} \to \mathbb{R}$, $F(z) = \log|z|$). But I don't need any of this here, since we can just write down a harmonic conjugate explicitly, as follows. Recall formula (2.4) for the Poisson kernel. So for real valued $g \in L^1(T)$, we can write

$$(P_r * g)(e^{ix}) = \operatorname{Re} \int_T \frac{1 + e^{-it}z}{1 - e^{-it}z}\, g(e^{it})\, d\sigma(t), \qquad z = re^{ix},$$

and now we can make a holomorphic function out of this by simply dropping the real part. Put differently, a harmonic conjugate of $P_r * g$ is given by $Q_r * g$, with

$$Q_r(e^{ix}) = \operatorname{Im} \frac{1+z}{1-z} = \frac{2r\sin x}{1 - 2r\cos x + r^2}.$$

We also call this function the *conjugate Poisson kernel,* for the obvious reasons.

Given a w as above, we can now form $H = \exp((P_r + iQ_r) * \log w)$. This is a holomorphic function on D with $|H| = \exp(P_r * \log w) \leq 1$, since, as we observed, $\log w \leq 0$ and thus $P_r * \log w \leq 0$ as well. Moreover, $|h(e^{ix})| = w(e^{ix})$ almost everywhere.

I now claim that $H \in H^\infty(D)$ is outer. To prove this, factor $H = QK$, with Q inner and K outer. H has no zeros, being an exponential of another function, so $Q(0) \neq 0$ also. Recall that $|q| = 1$, since q is inner, and if Q is not constant, then $|Q(0)| < 1$ by the maximum principle. Thus

$$\begin{aligned}
-\log|H(0)| &= -\int_T \log|h(e^{ix})|\, d\sigma(x) = -\int_T \log|k(e^{ix})|\, d\sigma(x) \\
&= \int_T \liminf_{r\to 1-}(-\log|K(re^{ix})|)\, d\sigma(x) \\
&\le \liminf_{r\to 1-}\left(-\int_T \log|K(re^{ix})|\, d\sigma(x)\right) \\
&= -\log|K(0)| < -\log|H(0)|.
\end{aligned}$$

Here I've used (6.1) for $\log|K|$, which works because this function is zero free, and Fatou's lemma for the first inequality. We have obtained a contradiction. So Q is in fact constant, and thus $H = QK = e^{i\alpha}K$ is outer, as claimed. Let's summarize what we just showed.

Lemma 6.1. *Let $w \in L^\infty(T)$ with $0 < w \le 1$ and $\log w \in L^1(T)$. Then $F = \exp((P_r + iQ_r) * \log w)$ is an outer function, $F \in H^\infty$, $|F| \le 1$.*

Now let's return to our original task. Assume, conversely, that an outer function $F \in H^\infty(D)$, $|F| \le 1$, is given. As we saw, we can then write $\log|F| = P_r * \mu$, and we wanted to show that in fact $\log|F| = P_r * \log|f|$. We also know, from the remarks following Theorem 2.3, that $d\mu = \log|f|\, d\sigma + d\nu$, with ν singular with respect to σ. The description of μ as the weak-$*$ limit of the measures $\log|f_r|\, d\sigma$ shows that μ, ν are negative measures (since $\log|f_r| \le 0$).

Define $G = \exp((P_r + iQ_r) * \nu)$. So $G \in H^\infty(D)$, $|G| \le 1$, and $\log|G| = P_r * \nu$. Moreover, the absolutely continuous part of the measure ν from the Poisson representation of $\log|G|$ is $\log|g|\, d\sigma$, but ν is purely singular, so $\log|g| = 0$. In other words, G is an inner function.

Since harmonic conjugates are essentially unique, we must have $F = e^{i\alpha}\exp((P_r + iQ_r) * \mu)$, and thus

$$F/G = e^{i\alpha}\exp((P_r + iQ_r) * \log|f|). \tag{6.2}$$

Since $\log|f| \le 0$, we see from this representation that $F/G \in H^\infty(D)$. So we can factor $F/G = PH$, with P inner, H outer, and this gives $F = (GP)H$, with GP inner also. But F was outer, by assumption, and these factorizations are essentially unique, so $GP \equiv e^{i\beta}$. This means that both G and $1/G$ are inner, so $|G| = 1$ on D, and thus $G \equiv e^{i\gamma}$ is constant itself. Now (6.2) shows that $\log|F| = P_r * \log|f|$, as claimed.

Let's again summarize. The restriction that our functions are bounded can be removed, and I'll state the result in this generality right away.

Theorem 6.2. *Let $F \in H^2(D)$ be an outer function. Then* $\log|f| \in L^1(T)$ *and* $\log|F| = P_r * \log|f|$, $\log F = i\alpha + (P_r + iQ_r) * \log|f|$.

Conversely, if a $w \in L^2(T)$ with $\log w \in L^1(T)$ *is given, then* $\log F = (P_r + iQ_r) * \log w$ *defines an outer function with* $|f| = w$.

In particular, since the inner factor doesn't affect the absolute value of a general $H^2(T)$ function, we have obtained a very satisfying description of the absolute values $|f|$ of functions $f \in H^2(T)$: a $w \ge 0$ satisfies $w = |f|$ for some $f \in H^2(T)$, $f \not\equiv 0$, if and only if $w \in L^2$ and $\log w \in L^1$. Since $\log w \le w^2$ for large w, the only potential problem with the integrability of $\log w$ occurs at the small values of w. So the condition $\log w \in L^1$ is best interpreted as a sharpened version of Theorem 4.3.

An interesting general feature of this result is that the functions $w = |f|$, $f \in H^p(T)$, satisfy no additional regularity condition other than $w \in L^p$ (if that even is a regularity condition). They are only restricted by not being allowed to vanish (or even become small) too frequently. On the other hand, f is the boundary value of the holomorphic function F. So such a boundary value can be rather irregular, for example discontinuous everywhere, even if F is bounded.

Another interesting insight contained in Theorem 6.2 is the fact that $|F|$ on T determines the whole function, up to a phase factor, if $F \in H^2(D)$ is outer. This is certainly not true if F is not outer. For example, $F(z) = z$ and $F = 1$ have the same absolute value on the boundary. In the next section, we will see that there is large supply of inner functions, even if only zero free functions are considered.

Theorem 6.2 describes the outer functions as those $F \in H^2$ for which $\log|F|$ is the Poisson integral of its boundary values $\log|f|$. This, and not our functional analytic condition from Definition 5.1, is usually taken as the defining property.

Exercise 6.1. Assuming Theorem 6.2, establish the analogous characterization of $|f|$ for $H^p(T)$ for arbitrary $1 \le p \le \infty$. More specifically, prove the following: $w \ge 0$ satisfies $w = |f|$ for some $f \in H^p(T)$, $f \not\equiv 0$, if and only if $w \in L^p(T)$ and $\log w \in L^1(T)$. *Suggestion:* First deal with the case $p = 1$, relying on Theorem 5.3. Then handle $H^p \subseteq H^1$ for general $p \ge 1$ by recalling that $H^p = H^1 \cap L^p$.

Proof of Theorem 6.2. Let $F \in H^2(D)$ be an outer function. We already proved the statements about F if F is, in addition, bounded. To remove this extra assumption, let's first establish that $\log|f| \in L^1(T)$ also in general. This we can do by essentially the original argument, in a slightly upgraded version: return to (6.1), which is clearly valid for any zero free holomorphic F, and split $\log|F| = \log^+|F| - \log^-|F|$

into its positive and negative parts (so for $x \in \mathbb{R}$ we define $x^+ = x$ if $x \geq 0$ and $x = 0$ otherwise, and similarly for x^-). Then

$$\begin{aligned}\int_T \log^- |f_r|\, d\sigma &= -\int_T \log|f_r|\, d\sigma + \int_T \log^+ |f_r|\, d\sigma \\ &= -\log|F(0)| + \int_T \log^+ |f_r|\, d\sigma.\end{aligned}$$

Now $\log^+ x > 0$ only if $x > 1$, so $\log^+ |f_r| \leq |f_r|^2$. This shows that the last integral stays bounded when we send $r \to 1-$, and thus Fatou's lemma still proves that $\log^- |f| \in L^1(T)$.

Let $w = \min\{1, 1/|f|\}$, so $0 < w \leq 1$, as above. Moreover, $\log w \leq |\log|f|| \in L^1(T)$. So we may use Lemma 6.1 to produce an outer function $G = \exp((P_r + iQ_r) * \log w)$, $|G| \leq 1$. The function FG also satisfies $|FG| \leq 1$ since $|fg| \leq |f|/|f| \leq 1$ on T. I now claim that it is also outer. To prove this, recall that F was outer, by assumption, so $M_f = H^2(T)$. In particular, we can find polynomials $P_n = \sum a_k e^{ikx}$ such that $P_n f \to 1$ in $H^2(T)$. Then also $P_n fg \to g$ in $H^2(T)$, since $|g| \leq 1$. This says that $g \in M_{fg}$, and thus $M_{fg} \supseteq M_g = H^2$. So fg is outer, as claimed. For future reference, let's state a general version of this step.

Lemma 6.3. *Let $f \in H^2$, let $g \in H^\infty$ be an outer function, and assume that $fg \in H^\infty$ also. Then f is outer if and only if fg is outer.*

Proof. We just proved one direction. To discuss the other, assume now that fg is outer, and factor $f = ph$, with p inner, h outer. Then $fg = p(gh)$, and here we know that $gh \in H^2$ is outer, by the direction already established. But fg is outer itself, so p is constant here and this shows that $f = ph$ is outer as well. □

Since $|FG| \leq 1$, we are back in the case already dealt with, and we know that $\log|FG| = P_r * \log|fg|$ is the Poisson integral of its boundary values. Since this is also true of $\log|G|$, which was in fact *defined* as a Poisson integral (of $\log w$), and $\log|FG| = \log|F| + \log|G|$ and similarly for $\log|fg|$, we conclude that $\log|F| = P_r * \log|f|$ also. The formula for $\log F$ is then an immediate consequence of this, by providing a harmonic conjugate.

The converse follows from *Jensen's inequality,* which says that if μ is a probability measure on a space X and $g : X \to \mathbb{R}$, $g \in L^1(X, \mu)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, then

$$\varphi\left(\int_X g(t)\, d\mu(t)\right) \leq \int_X \varphi(g(t))\, d\mu(t). \tag{6.3}$$

(In essence this is just restating the defining property of convex functions, namely that averages of values lie above the value at the average, in a continuous version, but of course a formal proof needs to be done more carefully.)

We'll apply this to $X = T$, $d\mu(t) = P_r(e^{i(x-t)})\, d\sigma(t)$, $\varphi(s) = e^s$, and $g(t) = \log w(e^{it})$. We obtain the inequality

$$e^{P_r * \log w} \le P_r * w.$$

Exercise 6.2. Explain this step in more detail, by writing out the integral implicit in $P_r * \log w$.

So $F = \exp((P_r + iQ_r) * \log w)$ satisfies $|F| = \exp(P_r * \log w) \le P_r * w$ and since $w \in L^2$, this latter function satisfies (2.7) for $p = 2$ and hence so does F. In other words, $F \in H^2(D)$, as claimed. To prove that F is outer, we use the same trick as above and reduce matters to the situation for bounded functions by introducing $G \in H^\infty$ as $\log |G| = P_r * \log v$, $v = \min\{1, 1/|f|\}$. (Or, what amounts to the same, $\log v = -\log^+ w$.) Then $|FG| \le 1$ and $\log |FG| = P_r * (-\log^- w)$, so FG is of the type discussed in Lemma 6.1 and hence this function is outer, and so is G, by the same argument. Now we finish the proof by referring to Lemma 6.3. □

7. Inner functions

It is now natural to round off our treatment by attempting to give a complex analytic description of inner functions also. I'll report quickly on this, but won't give any proofs.

One of the basic ideas of the previous section was to look at the Poisson representation of the harmonic function $\log |F|$. This we can also do for an inner function $S \in H^\infty(D)$ if S is zero free; if S has zeros, then of course $\log |S|$ isn't even defined everywhere on D. For a zero free S, we can write $\log |S| = P_r * \mu$, as in Theorem 2.3. Recall one more time that $d\mu$ is the weak-$*$ limit of the (signed) measures $\log |s_r|\, d\sigma$, and the absolutely continuous part of μ is $\log |s|\, d\sigma$. Since S is inner, $\log |s| = 0$. So the measure μ is purely singular, and negative, since $|s_r| \le 1$. So every zero free inner function S is of the form

$$S = e^{i\alpha} \exp((P_r + iQ_r) * \mu), \tag{7.1}$$

for some negative singular measure μ. Conversely, it is clear from the same arguments that for any such μ, (7.1) defines a zero free inner function, and in fact we observed this earlier, during the analysis leading up to Theorem 6.2. These functions are also called *singular inner functions.*

Of course, unlike outer functions, an inner function is not guaranteed to be zero free. For example, $F(z) = z$ is inner. We say that $z_n \in D$ satisfy the *Blaschke condition* if

$$\sum (1 - |z_n|) < \infty; \tag{7.2}$$

this is of course automatic if z_n is a finite sequence, and for an infinite sequence the condition insists on reasonably rapid approach to the boundary T of the z_n for large n.

Given $z_n \in D$ satisfying the Blaschke condition, we define the corresponding *Blaschke product* as

$$B(z) = z^k \prod \frac{z - z_n}{1 - \overline{z_n} z} \frac{|z_n|}{(-z_n)}.$$

(7.2) will imply that the infinite product (if it indeed is infinite) converges absolutely and locally uniformly on D and thus defines a holomorphic function $B : D \to \mathbb{C}$. Clearly, B has zeros at exactly the z_n. It can also be shown that $B \in H^\infty(D)$ and B is inner, that is, $|b(e^{ix})| = 1$ almost everywhere. (This is actually immediately plausible if you recognize the individual factors as Möbius transformations, so they map D back to itself bijectively, and they same is true for T.)

This two part list of inner functions now exhausts all possibilities in the sense that every inner function F is of the form $F = BS$, with B a Blaschke product and S a singular inner function. As an interesting corollary of this description we obtain a characterization of the zero sets of functions $F \in H^p(D)$ as exactly the sequences satisfying the Blaschke condition (7.2).

A general $F \in H^2(D)$ can now be written as $F = BSG$, with B a Blaschke product, S a singular inner function, and G outer. A good summary of what is going on here goes as follows: given an $F \in H^2(D)$, first of all divide out its zeros with the help of a Blaschke product, so form F/B. This is a zero free holomorphic function, so $\log|F/B|$ is harmonic. Moreover, it has a Poisson representation $\log|F/B| = P_r * \mu$. In general, μ will have an absolutely continuous part $\log|f/b|\, d\sigma = \log|f|\, d\sigma$, and a (negative) singular part ν. The absolutely continuous part is responsible for the outer factor $\log|G| = P_r * \log|f|$, while the singular part produces the singular inner factor $\log|S| = P_r * \nu$.