Math 3333, Quiz VI, March 11, 2008

1. Let $V = M_{22}$ and $W =$ subset of all upper triangular matrices. Is W a vector subspace ? Explain.

We see that W consists of 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with real numbers a, b, d. Let $\mathbf{w}_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ and c any real number. Then

$$\mathbf{w}_1 \oplus \mathbf{w}_2 = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix} \text{ which lies in } W \text{ and}$$

$$c \odot \mathbf{w}_1 = c \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} = \begin{bmatrix} ca_1 & cb_1 \\ 0 & cd_1 \end{bmatrix} \text{ which lies in } W.$$

Hence W is a vector subspace.

2. Let $V =$ all non-zero real numbers. Let $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $c \odot \mathbf{u} = c\mathbf{u}$. Is V a vector space ? Explain.

If we take $c = 0$ and any $\mathbf{u}$ in V then $c \odot \mathbf{u} = c\mathbf{u} = 0\mathbf{u} = 0$. But 0 does not lie in V and hence V is not closed under the $\odot$ operation. Hence V is not a vector space.

Note that all the properties $(a), (1), (2), (3), (4)$ are satisfied. In particular, the role of the $\mathbf{0}$ vector is played by the real number 1 and if $\mathbf{u}$ is any real number then the vector $-\mathbf{u}$ will be given by $1/u$. (Check that these choices satisfy properties (3) and (4)).

3. Let $V = P_2$ and $W =$ all polynomials $p(t)$ in V such that $p(0) = 0$. Is W a vector subspace ? Explain.

We see that W consists of those polynomials $p(t) = a_2t^2 + a_1t + a_0$ such that $p(0) = 0$. This implies that $a_0 = 0$, i.e. $p(t) = a_2t^2 + a_1t$. Let $q(t) = b_2t^2 + b_1t$ and c be any real number. Then

$$p(t) \oplus q(t) = (a_2t^2 + a_1t) + (b_2t^2 + b_1t) = (a_2 + b_2)t^2 + (a_1 + b_1)t \text{ which lies in } W \text{ and}$$

$$c \odot p(t) = c(a_2t^2 + a_1t) = ca_2t^2 + ca_1t \text{ which lies in } W.$$

Hence W is a vector subspace.