Math 3333, Quiz III, February 5, 2008

1. (5 Points) Let A be a $n \times n$ skew-symmetric matrix. Show that if k is any even integer, then the matrix A^k is symmetric.

   This problem is similar to Problem 28 of section 1.5 in homework 2

   Since A is skew-symmetric, we have $A^T = -A$. We have to show that A^k is symmetric, i.e., $(A^k)^T = A^k$. We have
   $$(A^k)^T = (A^T)^k = (-A)^k = (-1)^k A^k = A^k \text{ since } k \text{ is even.}$$

2. (7 Points) Consider the linear system $(AB)^T\mathbf{x} = \mathbf{b}$, with A, B invertible matrices. Find the solution to the linear system if $A^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 0 & -4 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

   This problem is similar to problem 38 of Section 1.5 of homework 2 and problem 2 of Quiz II

   We have
   $$(AB)^T\mathbf{x} = \mathbf{b} \Rightarrow B^TA^T\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = (B^TA^T)^{-1}\mathbf{b} \Rightarrow \mathbf{x} = (A^T)^{-1}(B^T)^{-1}\mathbf{b} \Rightarrow \mathbf{x} = (A^{-1})^T(B^{-1})^T\mathbf{b}.$$
   Hence
   $$\mathbf{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}^T \begin{bmatrix} 0 & -4 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -12 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -26 \end{bmatrix}.$$

3. (8 Points) Consider the matrix transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(\mathbf{u}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{u}$. Determine whether $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}$ are in the range of f.

   This problem is similar to Problems 9 and 13 of section 1.6 in homework 2

   From the definition of f we see that
   $$f\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_1 \end{bmatrix}.$$
   We know that a vector $\mathbf{w}$ in $\mathbb{R}^3$ is in the range of f if we can find a vector $\mathbf{u}$ in $\mathbb{R}^2$ such that $f(\mathbf{u}) = \mathbf{w}$. For $\mathbf{w}_1$ and $\mathbf{w}_2$ we get the equations
   $$f(\mathbf{u}) = \mathbf{w}_1 \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \text{ inconsistent,} \quad f(\mathbf{u}) = \mathbf{w}_2 \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} \Rightarrow u_1 = 4, u_2 = -2.$$
   Hence, $\mathbf{w}_1$ is not in the range of f and $\mathbf{w}_2$ is in the range of f.