MATH 3333

Midterm I

February 21, 2008

<u>Name</u> :

<u>I.D. no.</u> :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.

- Show as much work as possible. Answers without explanation will not receive any credit.

- If you preform any row or column operations in a problem, record them using standard notations.

- Best of Luck.

i) Let $A = \begin{bmatrix} 3 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix}$.

a) (18 Points) Find $\det(A)$.

There are various elementary row or column operations that you can apply to get the above matrix into an upper or lower triangular form. Probably the easiest way is as follows :

$$\det\left(\begin{bmatrix} 3 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix}\right) = (-1)(-1)\det\left(\begin{bmatrix} 3 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix}_{\substack{\mathbf{r_1}\leftrightarrow\mathbf{r_3} \\ \mathbf{r_2}\leftrightarrow\mathbf{r_4}}}\right)$$

$$= (-1)(-1)\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}\right) = (-1)(-1)(1)(1)(1)(1) = \boxed{1}$$

b) (2 Points) Is the matrix A invertible ?

The matrix A is invertible because $\det(A) \neq 0$.

c) (20 Points) Find A^{-1}, if it exists.

To find A^{-1}, one has to consider the matrix $[A|I_4]$ and apply elementary row operations to obtain $[I_4|A^{-1}]$.

$$\left(\begin{array}{cccc|cccc} 3 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathbf{r}_1 \leftrightarrow \mathbf{r}_3, \mathbf{r}_2 \leftrightarrow \mathbf{r}_4 \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$-5\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2, -3\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -5 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3, 2\mathbf{r}_2 + \mathbf{r}_4 \to \mathbf{r}_4 \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -8 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -10 & 2 \end{array}\right)$$

Hence, we have

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 \\ 1 & 0 & -8 & 1 \\ 0 & 1 & -10 & 2 \end{bmatrix}.$$

Note that you are not allowed to mix column and row operations while finding A^{-1}. Also, if you started with $[I_4|A]$, then you are only allowed to apply elementary column operations to get $[A^{-1}|I_4]$.

ii) (20 Points) Find a 3×1 matrix $\mathbf{x}$ with entries not all zero such that

$$A\mathbf{x} = 3\mathbf{x}, \qquad \text{where } A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}.$$

We have

$$A\mathbf{x} - 3\mathbf{x} = \mathbf{0} \Rightarrow A\mathbf{x} - 3I_3\mathbf{x} = \mathbf{0} \Rightarrow (A - 3I_3)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} -3 & 0 & 3 \\ 1 & -3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

Now consider the augmented matrix of the above homogeneous linear system.

$$\left(\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 1 & -3 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \qquad \left(\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ -3 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \qquad \left(\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 \qquad \left(\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right)$$

$$9\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3 \qquad \left(\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The last matrix is in row echelon form and the corresponding linear system is

$$x_1 - 3x_2 - x_3 = 0 \qquad x_2 = 0.$$

This gives us

$$x_2 = 0 \text{ and } x_1 = x_3.$$

Let $x_3 = r$, any real number, then any solution of the equation $A\mathbf{x} = 3\mathbf{x}$ is given by

$$\mathbf{x} = \begin{bmatrix} r \\ 0 \\ r \end{bmatrix}.$$

Instead of using the augmented matrix one can directly look at the linear system

$$-3x_1 + 3x_3 = 0, \qquad x_1 - 3x_2 - x_3 = 0, \qquad x_2 = 0$$

which gives the same solutions.

iii) (20 Points) Are the following statements true or false ? Explain your answers.

a) The permutation 635142 of $\{1, 2, 3, 4, 5, 6\}$ is an even permutation.

$\boxed{\text{FALSE}}$ There are 11 inversions : 6 precedes $3, 5, 1, 4, 2$ (5 inversions), 3 precedes $1, 2$ (2 inversions), 5 precedes $1, 4, 2$ (3 inversions), 4 precedes 2 (1 inversion). Hence the permutation is odd.

b) Any 3×3 matrix A with real matrix entries satisfying $A^3 = -3A$ is singular.

$\boxed{\text{TRUE}}$ Taking determinants of both sides of $A^3 = -3A$, we get

$$\begin{aligned} & \det(A^3) = \det(-3A) \Rightarrow \det(A)^3 = \det(-3I_3)\det(A) \\ \Rightarrow \quad & \det(A)^3 = -27\det(A) \Rightarrow \det(A)^3 + 27\det(A) = 0 \\ \Rightarrow \quad & \det(A)(\det(A)^2 + 27) = 0 \Rightarrow \det(A) = 0 \text{ or } \det(A)^2 + 27 = 0. \end{aligned}$$

We cannot have $\det(A)^2 + 27 = 0 \Rightarrow \det(A)^2 = -27$ because $\det(A)$ is a real number and the square of a real number cannot be negative. Hence

$$\det(A) = 0 \Rightarrow \text{ The matrix } A \text{ is singular.}$$

c) Suppose $\mathbf{x}_1$ is a solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_1 + \mathbf{x}_2$ is a solution of $A\mathbf{x} = \mathbf{b}$.

$\boxed{\text{TRUE}}$ Since $\mathbf{x}_1$ is a solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$, we have

$$A\mathbf{x}_1 = \mathbf{b} \text{ and } A\mathbf{x}_2 = \mathbf{0}.$$

We have to show that $\mathbf{x}_1 + \mathbf{x}_2$ is also a solution of $A\mathbf{x} = \mathbf{b}$. For this, we start with

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

as required.

iv) (10 Points) Let A be a $n \times n$ matrix. If $A^T = A^{-1}$, then show that $\det(A) = \pm 1$.

Taking determinant of both sides of $A^T = A^{-1}$ we get

$$\det(A^T) = \det(A^{-1}).$$

Now using the fact that $\det(A^T) = \det(A)$ and $\det(A^{-1}) = 1/\det(A)$, we get

$$\det(A) = \frac{1}{\det A} \Rightarrow \det(A)^2 = 1 \Rightarrow \det(A) = \pm 1,$$

as required.

v) (10 Points) Give a geometric description of the matrix transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\mathbf{u}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$.

Let $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$f(\mathbf{u}) = f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

This shows us that f is a projection onto the x-axis.

vi) (Extra credit 10 Points) Let $A = [a_{ij}]$ be a $n \times n$ matrix. Define the trace of the matrix A to be the sum of the entries in the main diagonal, i.e., $Trace(A) := \sum_{i=1}^{n} a_{ii}$. Show that

$$Trace(A^T A) \geq 0.$$

Let $B = A^T A = [b_{ij}]$. Then, by the above definition

$$Trace(A^T A) = Trace(B) = \sum_{i=1}^{n} b_{ii}.$$

Now, using the formula for matrix multiplication, we have

$$b_{ii} = \sum_{k=1}^{n} a_{ik}^T a_{ki} = \sum_{k=1}^{n} a_{ki} a_{ki} = \sum_{k=1}^{n} a_{ki}^2 \geq 0 \Rightarrow b_{ii} \geq 0 \text{ for every } i = 1, \cdots, n$$

Hence, we get

$$Trace(A^T A) = \sum_{i=1}^{n} b_{ii} \geq 0,$$

as required.

Note that the Trace function is different from the determinant in the sense that $Trace(AB) \neq Trace(A)Trace(B)$.