# Solution to HW 8

## Section 4.3

16) $V = P_2$

(a) $W = \{ a_2t^2 + a_1t + a_0 : a_1 = 0,\ a_0 = 0 \}$ i.e. $W = \{ a_2t^2 \}$

Let $\underline{w}_1 = a_2t^2$, $\underline{w}_2 = b_2t^2$, $c$ real no.

$\underline{w}_1 \oplus \underline{w}_2 = a_2t^2 + b_2t^2 = (a_2+b_2)t^2$ is in $W$

$c \odot \underline{w}_1 = c(a_2t^2) = (ca_2)t^2$ is in $W$

$\Rightarrow$ $W$ is a subspace of $P_2$

(b) $W = \{ a_2t^2 + a_1t + a_0 : a_1 = 2a_0 \}$ i.e. $W = \{ a_2t^2 + (2a_0)t + a_0 \}$

Let $\underline{w}_1 = a_2t^2 + (2a_0)t + a_0$, $\underline{w}_2 = b_2t^2 + (2b_0)t + b_0$, $c$ real no.

$\underline{w}_1 \oplus \underline{w}_2 = (a_2+b_2)t^2 + (2a_0+2b_0)t + (a_0+b_0)$

$= (a_2+b_2)t^2 + [2(a_0+b_0)]t + (a_0+b_0)$ is in $W$

$c \odot \underline{w}_1 = c(a_2t^2 + 2a_0t + a_0) = (ca_2)t^2 + 2(ca_0)t + (ca_0)$ is in $W$.

$\Rightarrow$ $W$ is a subspace of $P_2$.

(c) $W = \{ a_2t^2 + a_1t + a_0 : a_2 + a_1 + a_0 = 2 \}$

Let $\underline{w}_1 = a_2t^2 + a_1t + a_0$ be in $W$ i.e. $a_2 + a_1 + a_0 = 2$

& $c = 2$ then $c \odot \underline{w}_1 = (ca_2)t^2 + (ca_1)t + (ca_0)$

$= (2a_2)t^2 + (2a_1)t + (2a_0)$

but

$2a_2 + 2a_1 + 2a_0 = 2(a_2 + a_1 + a_0) = 2 \cdot 2 = 4 \neq 2$

$\Rightarrow$ $W$ is <u>not</u> a subspace of $P_2$.

(18) $V = M_{nn}$

(a) $W$ = subset of singular matrices

i.e. $W = \{ A \text{ in } M_{nn} : \det(A) = 0 \}$.

let $A_1 = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & \ddots \; 1 \end{bmatrix}$ $\qquad A_2 = \begin{bmatrix} 1 & 1 & & & \\ 1 & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$

By expanding along $1^{st}$ row (or column) we can see that $\det(A_1) = 0$ &

$\det(A_2) = 0$. But $A_1 + A_2 = \begin{bmatrix} 2 & & & \\ & 2 & & 0 \\ & & 2 & \\ & 0 & & \ddots \; 2 \end{bmatrix}$ & $\det(A_1 + A_2) = \underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}}$

$= 2^n \neq 0$

i.e. $A_1 + A_2$ not in $W$.

$\Rightarrow$ $W$ is <u>not</u> a subspace

(b) $W$ = subset of all upper triangular matrices

i.e. $W = \{ A = [a_{ij}] : a_{ij} = 0 \text{ if } i > j \}$

let $A = [a_{ij}]$ in $W$ & $B = [b_{ij}]$ in $W$ & $c$ a real no.

i.e. $a_{ij} = 0$ if $i < j$ $\qquad b_{ij} = 0$ if $i < j$

$A \oplus B = C = [c_{ij}] = [a_{ij} + b_{ij}]$ $\quad$ if $i < j$ $\quad c_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0$

$\Rightarrow$ $C = [c_{ij}]$ is upper triangular $\Rightarrow$ $C$ is in $W$

$cA = [c a_{ij}]$ $\quad$ for $i < j$ the $(i,j)^{th}$ entry is $c a_{ij} = c \cdot 0 = 0$

$\Rightarrow$ $cA$ is still upper triangular $\Rightarrow$ $cA$ is in $W$

$\Rightarrow$ $W$ is a subspace.

(c) $W$ = set of all skew-symmetric matrices

i.e. $W = \{ A \text{ in } M_{nn} : A^T = -A \}$

Let $A, B$ be in $W$ i.e. $A^T = -A$ & $B^T = -B$

& $c$ a real no.

$A \oplus B = A + B$ ; $(A+B)^T = A^T + B^T = (-A) + (-B) = -(A+B)$

$\Rightarrow (A+B)^T = -(A+B) \Rightarrow A+B$ is in $W$.

$c \odot A = cA$ ; $(cA)^T = cA^T = c(-A) = -(cA)$

$\Rightarrow (cA)^T = -(cA) \Rightarrow cA$ is in $W$

$\Rightarrow W$ is a subspace.

24) $A$ non-singular matrix.

Null space of $A$ = { sol$^n$s to the homogeneous linear system $A\underline{x} = \underline{0}$ }

If $A$ is non-singular $\Rightarrow A^{-1}$ exists

$\Rightarrow A^{-1}(A\underline{x}) = A^{-1}\underline{0} = \underline{x} = \underline{0}$

$\Rightarrow$ Null space of $A = \{\underline{0}\}$.

31) $W = \left\{ \begin{bmatrix} a & b & c \\ a & 0 & 0 \end{bmatrix} : c = a+b \right\}$ is a subspace of $M_{23}$

Let $\underline{w}$ be in $W$. Then $\underline{w} = \begin{bmatrix} a & b & c \\ a & 0 & 0 \end{bmatrix}$ with $c = a+b$

$\Rightarrow \underline{w} = \begin{bmatrix} a & b & a+b \\ a & 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \underline{w} = a\,\underline{w}_1 + b\,\underline{w}_2$ with $\underline{w}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ & $\underline{w}_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ every vector in $W$ can be written as a linear combination of $\underline{w}_1$ & $\underline{w}_2$.

(34) $V = R_4$ $\quad \underline{v}_1 = [1\ 2\ 1\ 0]$, $\underline{v}_2 = [4\ 1\ -2\ 3]$, $\underline{v}_3 = [1\ 2\ 6\ -5]$, $\underline{v}_4 = [-2\ 3\ -1\ 2]$.

(a) $\underline{v} = [3\ 6\ 3\ 0]$

Let $\underline{v} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 + a_4\underline{v}_4$ This gives the linear system

$$\begin{bmatrix} 1 & 4 & 1 & -2 \\ 2 & 1 & 2 & 3 \\ 1 & -2 & 6 & -1 \\ 0 & 3 & -5 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \\ 0 \end{bmatrix}$$

one $sol^n$ is $a_1 = 3,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0$ (work out the details !!.)

$\Rightarrow \underline{v} = 3\,\underline{v}_1 + 0\,\underline{v}_2 + 0\,\underline{v}_3 + 0\,\underline{v}_4$ $\Rightarrow \underline{v}$ is a linear combination of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$.

b) Let $\underline{v} = [1\ 0\ 0\ 0]$ Let $\underline{v} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 + a_4\underline{v}_4$

This gives the linear system $\begin{bmatrix} 1 & 4 & 1 & -2 \\ 2 & 1 & 2 & 3 \\ 1 & -2 & 6 & -1 \\ 0 & 3 & -5 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

no $\underline{sol^n}$ to the linear system (check the details !!)

$\Rightarrow \underline{v}$ is not a linear combination of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$

(c) $\underline{v} = [3 \ 6 \ -2 \ 5]$ As before we get a linear system

$$\begin{bmatrix} 1 & 4 & 1 & -2 \\ 2 & 1 & 2 & 3 \\ 1 & -2 & 6 & -1 \\ 0 & 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -2 \\ 5 \end{bmatrix}$$

one $sol^n$ is $a_1 = 4, a_2 = 0, a_3 = -1, a_4 = 0$

(check !!)

$\Rightarrow \underline{v} = 4\underline{v}_1 + 0\underline{v}_2 + (-1)\underline{v}_3 + 0\underline{v}_4 \Rightarrow \underline{v}$ ~~then~~ is a linear combination of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$

d) $\underline{v} = [0 \ 0 \ 0 \ 1]$ We get the linear system

$$\begin{bmatrix} 1 & 4 & 1 & -2 \\ 2 & 1 & 2 & 3 \\ 1 & -2 & 6 & -1 \\ 0 & 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

No $\underline{sol^n}$. (check !!)

$\Rightarrow \underline{v}$ is not a linear combination of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$.

Section 4.4 :

3 $V = P_2$ $S = \{ p_1(t) = t^2 + 2t + 1, \ p_2(t) = t^2 + 3, \ p_3(t) = t - 1 \}$

In each of the cases, we write

$p(t) = a_1 p_1(t) + a_2 p_2(t) + a_3 p_3(t)$ & get a linear

system

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{(*)}$$

if $p(t) = at^2 + bt + c$.

a) $p(t) = t^2 + t + 2$ $\quad a = 1, b = 1, c = 2$

Solving the above linear system (*) we get one $sol^n$ $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}, a_3 = 0$

(check !!)

$\Rightarrow p(t) = \frac{1}{2} p_1(t) + \frac{1}{2} p_2(t) + 0 p_3(t) \Rightarrow p(t)$ lies in span(S).

(b) $p(t) = 2t^2 + 2t + 3 \Rightarrow a = 2,\ b = 2,\ c = 3$ in the linear system $(*)$

No ~~st~~ $\underline{\text{sol}}^n$ to the linear system $(*)$ (Check!!)

$\Rightarrow p(t)$ not in $\mathrm{Span}(S)$.

(c) $p(t) = -t^2 + t - 4 \Rightarrow a = -1,\ b = 1,\ c = -4$ in the linear system $(*)$

$a_1 = \frac{1}{2},\ a_2 = -\frac{3}{2},\ a_3 = 0$ is a $\text{sol}^n$. (Check!!)

$\Rightarrow p(t) = \frac{1}{2} p_1(t) + \left(-\frac{3}{2}\right) p_2(t) + (0)\, p_3(t) \Rightarrow p(t)$ lies in $\mathrm{span}(S)$.

(d) $p(t) = -2t^2 + 3t + 1 \Rightarrow a = -2,\ b = 3,\ c = 1$ in the linear system $(*)$

No $\underline{\text{sol}}^n$ to the linear system $(*)$ (Check!!)

$\Rightarrow p(t)$ is not in the $\mathrm{span}(S)$.

⑥ $V = R^4$

a) $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ Let $\underline{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be any vector in $R^4$

Want to see if $\underline{v} = a_1 \underline{v}_1 + a_2 \underline{v}_2$. This gives a linear system

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ -1 & 1 & b \\ 2 & 1 & c \\ 0 & 1 & d \end{array}\right] \xrightarrow[-2r_1 + r_3 \to r_3]{r_1 + r_2 \to r_2} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & a+b \\ 0 & 1 & c-2a \\ 0 & 1 & d \end{array}\right] \xrightarrow[-r_2 + r_4 \to r_4]{-r_2 + r_3 \to r_3} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & a+b \\ 0 & 0 & c-3a-b \\ 0 & 0 & d-a-b \end{array}\right]$$

The system has a solution only if $c - 3a - b = 0$ and $d - a - b = 0$

This does not include all vectors of $R^4$. Hence $\boxed{\mathrm{Span}\{\underline{v}_1, \underline{v}_2\} \neq R^4}$.

(b) $\underline{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Let $\underline{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be any element in $R^4$

Want to see if $\underline{v} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3$ This gives a linear system:

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & a \\ 2 & 2 & 0 & b \\ 1 & -1 & 0 & c \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & c \\ 2 & 2 & 0 & b \\ 3 & 1 & 0 & a \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow[-3r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \left[\begin{array}{cccc} 1 & -1 & 0 & c \\ 0 & 3 & 0 & b-2c \\ 0 & 4 & 0 & a-3c \\ 0 & 0 & 1 & d \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & c \\ 0 & 1 & 0 & \frac{b-2c}{3} \\ 0 & 4 & 0 & a-3c \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow{-4r_2 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & c \\ 0 & 1 & 0 & \frac{b-2c}{3} \\ 0 & 0 & 0 & a - \frac{4b}{3} - \frac{c}{3} \\ 0 & 0 & 1 & d \end{array}\right]$$

This has a $sol^n$ only if $a - \frac{4b}{3} - \frac{c}{3} = 0$. This does not include all vectors in $R^4$ $\Rightarrow$ $\text{span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\} \neq R^4$.

(c) $\underline{v}_1 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \end{bmatrix}$, $\underline{v}_4 = \begin{bmatrix} 5 \\ 6 \\ -3 \\ 2 \end{bmatrix}$, $\underline{v}_5 = \begin{bmatrix} 0 \\ 4 \\ -2 \\ -1 \end{bmatrix}$.

Let $\underline{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be any element of $R^4$. Want to see if $\underline{v} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 + a_4\underline{v}_4 + a_5\underline{v}_5$

This gives a linear system

$$\left[\begin{array}{ccccc|c} 3 & 4 & 3 & 5 & 0 & a \\ 2 & 0 & 2 & 6 & 4 & b \\ -1 & 0 & -1 & -3 & -2 & c \\ 2 & 2 & 2 & 2 & -1 & d \end{array}\right] \xrightarrow{2r_3 + r_2 \to r_2} \left[\begin{array}{ccccc|c} 3 & 4 & 3 & 5 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & b+2c \\ -1 & 0 & -1 & -3 & -2 & c \\ 2 & 2 & 2 & 2 & -1 & d \end{array}\right]$$

This has a $sol^n$ only if $b + 2c = 0$

This does not include all vectors in $R^4$. $\Rightarrow \text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5\} \neq R^4$.

(d) $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\underline{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ Let $\underline{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be any vector in $R^4$.

Want to see if $\underline{v} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 + a_4\underline{v}_4$. This gives a linear system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 1 & 2 & 0 & 1 & b \\ 0 & -1 & 1 & 2 & c \\ 0 & 1 & 1 & -1 & d \end{array}\right] \xrightarrow{-r_1+r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 0 & 1 & 0 & -1 & b-a \\ 0 & -1 & 1 & 2 & c \\ 0 & -1 & 1 & -1 & d \end{array}\right] \xrightarrow[r_2+r_4 \to r_4]{r_2+r_3 \to r_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 0 & 1 & 0 & -1 & b-a \\ 0 & 0 & 1 & 1 & c+b-a \\ 0 & 0 & 1 & -2 & d+b-a \end{array}\right]$$

$$\xrightarrow{-r_3+r_4 \to r_4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 0 & 1 & 0 & -1 & b-a \\ 0 & 0 & 1 & 1 & c+b-a \\ 0 & 0 & 0 & -3 & d-c \end{array}\right] \Rightarrow a_4 = \frac{d-c}{3}, \quad a_3 = \frac{4c}{3} + b - a - \frac{d}{3},$$

$$a_2 = b - a + \frac{d-c}{3}, \quad a_1 = 2a - b + c - d$$

$$\Rightarrow \underline{v} = (2a-b+c-d)\,\underline{v}_1 + \left(b-a+\frac{d-c}{3}\right)\underline{v}_2 + \left(\frac{4c}{3}+b-a-\frac{d}{3}\right)\underline{v}_3 + \left(\frac{d-c}{3}\right)\underline{v}_4$$

$\Rightarrow \underline{v}$ lies in span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$

$\Rightarrow$ span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\} = R^4$.

(9) $V = P_3$ $\quad S = \{\underline{v}_1 = t^3+2t+1,\ \underline{v}_2 = t^2-t+2,\ \underline{v}_3 = t^3+2,\ \underline{v}_4 = -t^3+t^2-5t+2\}$

Let $\underline{v} = at^3+bt^2+ct+d$ be any vector in $P_3$. Want to see if

$\underline{v} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 + a_4\underline{v}_4$. This gives a linear system

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ \boxed{2} & -1 & 0 & -5 & c \\ \boxed{1} & 2 & 2 & 2 & d \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & \boxed{-1} & -2 & -3 & c-2a \\ 0 & \boxed{2} & 1 & 3 & d-a \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & \boxed{-2} & -2 & c+b-2a \\ 0 & 0 & 1 & 1 & d-a-2b \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & c-3b-4a+2d \\ 0 & 0 & 1 & 1 & d-a-2b \end{array}\right]$$

has a sol$^n$ only if $c-3b-4a+2d = 0$

This does not include all vectors in $P_3$

$\Rightarrow$ span$(S) \neq P_3$.

11 $A\underline{x} = 0$ with $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ \boxed{1} & 2 & 3 & 1 & 0 \\ \boxed{2} & 1 & 3 & 1 & 0 \\ \boxed{1} & 1 & 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & \boxed{2} & 2 & 1 & 0 \\ 0 & \boxed{1} & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row echelon form

$\Rightarrow$ $x_1 + x_3 = 0$

$x_2 + x_3 + x_4 = 0$

$\boxed{x_4 = 0}$

Let $x_3 = r$ any real no.

$\Rightarrow x_2 = -r$ & $x_1 = -r$

$\Rightarrow$ Sol$^n$ to $A\underline{x} = \underline{0}$ is given by

$$\begin{bmatrix} -r \\ -r \\ r \\ 0 \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$ spans the solution space of $A\underline{x} = \underline{0}$.

15 $V = M_{33}$, $W = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \right\}$

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{v}_1} + b\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{v}_2} + c\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{v}_3} + d\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\underline{v}_4} + e\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{v}_5}$$

$\Rightarrow \text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5\} = W$.

## Section 4.5

1 $V = R^3$ $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix} \right\}$ S has 3 vectors from $R^3$

Consider $A = \begin{bmatrix} 2 & 3 & 10 \\ 1 & -1 & 0 \\ 3 & 2 & 10 \end{bmatrix}$ Then $\det(A) = 0$ $\Rightarrow$ S is linearly dependent.

(3) $V = \mathbb{R}^4$ $\quad S = \left\{ \underset{\underline{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}}, \underset{\underline{v}_2}{\begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}}, \underset{\underline{v}_3}{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}} \right\}$

Want to see if $a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3 = \underline{0}$. This gives a homogeneous linear system

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ \boxed{2} & 3 & 0 & 0 \\ \boxed{1} & 1 & 1 & 0 \\ \boxed{-1} & 0 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -5 & -4 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 4 & 5 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 4/5 & 0 \\ 0 & \boxed{-3} & -1 & 0 \\ 0 & \boxed{4} & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 4/5 & 0 \\ 0 & 0 & 7/5 & 0 \\ 0 & 0 & 9/5 & 0 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 4/5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{a_1 = 0, \ a_2 = 0, \ a_3 = 0} \text{ only sol}^n$$

Row echelon form

$\Rightarrow$ S is linearly independent.