Solution to H.W. 7

Section 4.2:

② $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : abcd = 0 \right\}$

(a) No. Let $\underline{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ & $\underline{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in V $\quad \underline{v}_1 \oplus \underline{v}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ not in V.

(b) Yes. Let c real no. $\underline{v} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ in V (i.e. $a_1 b_1 c_1 d_1 = 0$)

$c \odot \underline{v} = c \odot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix}$ Now $(ca_1)(cb_1)(cc_1)(cd_1) = c^4 \underbrace{(a_1 b_1 c_1 d_1)}_{=0}$ $= 0$.

$\Rightarrow c \odot \underline{v}$ in V.

(c) $\underline{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies $\underline{0} + \underline{v} = \underline{v} + \underline{0} = \underline{v}$ for all $\underline{v}$ in V

(d) Yes. Let $\underline{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in V (i.e. $abcd = 0$)

Then negative of $\underline{v}$: $-\underline{v} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ is also in V since $(-a)(-b)(-c)(-d) = (abcd) = 0$.

(e) No. not closed under addition

③ $V = \left\{ \begin{bmatrix} a & b \\ 2b & d \end{bmatrix} \quad a, b, d \text{ real nos.} \right\}$

(a) Yes $\quad \underline{v}_1 = \begin{bmatrix} a_1 & b_1 \\ 2b_1 & d_1 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} a_2 & b_2 \\ 2b_2 & d_2 \end{bmatrix}$ in V

$\underline{v}_1 \oplus \underline{v}_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 2b_1 + 2b_2 & d_1 + d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 2(b_1 + b_2) & d_1 + d_2 \end{bmatrix}$ also in V.

b) Yes $\quad$ c a real no. & $\underline{v} = \begin{bmatrix} a & b \\ 2b & d \end{bmatrix}$ in V

$c \odot \underline{v} = c \begin{bmatrix} a & b \\ 2b & d \end{bmatrix} = \begin{bmatrix} ac & bc \\ 2bc & dc \end{bmatrix} = \begin{bmatrix} ac & bc \\ 2(bc) & dc \end{bmatrix}$ also in V

(c) $\underline{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in V & satisfies $\underline{0} \oplus \underline{v} = \underline{v} \oplus \underline{0} = \underline{v}$ for all $\underline{v}$ in V

d) Yes. Let $\underline{v} = \begin{bmatrix} a & b \\ 2b & d \end{bmatrix}$ in V then $-\underline{v} = \begin{bmatrix} -a & -b \\ 2(-b) & -d \end{bmatrix}$ also in V

(e) Yes, One can check that all the properties (a) 1, 2, 3, 4 (b) 5, 6, 7, 8 are satisfied. You have to check them all.

(7) V = positive real nos $\quad \oplus$ usual addition $\quad \odot$ usual mult.

~~(a) sat~~ (a) (1) (2) <u>satisfied</u>

(3), (4) <u>not satisfied</u> since there is no suitable $\underline{0}$ vector & hence no negative vectors

(b) <u>not satisfied</u> : $(-1)\odot(20) = -20$ not in V.

(5), (6) (7) <u>cannot be applied</u>

(8) <u>satisfied</u> since $1 \odot \underline{v} = 1(v) = v$ ✓

(10) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \ x \le 0 \right\}$

$\underline{0}$ vector is $\underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in V.

(a) 1, 2, 3, are <u>satisfied</u>

(4) <u>not satisfied</u> since $-\underline{v}$ would have to be $\begin{bmatrix} -x \\ -y \end{bmatrix}$

but then $-x \not\le 0$.

(b) <u>not satisfied</u> $\quad -2 \odot \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$ not in V

(5), (6), (7) <u>not satisfied</u>

(8) <u>satisfied</u> because $1 \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ ✓

$\underline{v} \oplus \underline{u} = \underline{v}\,\underline{u} - 1, \ c \odot \underline{v} = \underline{v}$.

(16) V = positive real nos.

Not a vector space because V not closed under $\oplus$

$$\tfrac{1}{2} \oplus \tfrac{1}{2} = \tfrac{1}{2}\cdot\tfrac{1}{2} - 1 = -\tfrac{3}{4} \text{ not in } V.$$

(24) Prove $\cancel{v} + \underline{u} \oplus \underline{v} = \underline{u} \oplus \underline{w} \Rightarrow \underline{v} = \underline{w}$

By (4) $-\underline{u}$ exists:

$$-\underline{u} \oplus (\underline{u} \oplus \underline{v}) = -\underline{u} \oplus (\underline{u} \oplus \underline{w})$$

By (2) $\Rightarrow$ $(-\underline{u} \oplus \underline{u}) \oplus \underline{v} = (-\underline{u} \oplus \underline{u}) \oplus \underline{w}$

By (4) $\Rightarrow$ $\underline{0} \oplus \underline{v} = \underline{0} \oplus \underline{w}$

By (3) $\Rightarrow$ $\underline{v} = \underline{w}$ as required

(25) If $\underline{u} \neq 0$, prove that $a \odot \underline{u} = b \odot \underline{u} \Rightarrow a = b$

By (4) $-b \odot \underline{u}$ exists:

$$a \odot \underline{u} \oplus (-b \odot \underline{u}) = b \odot \underline{u} \oplus (-b \odot \underline{u})$$

By (4) & (6) $(a-b) \odot \underline{u} = \underline{0}$

If $\boxed{a \neq b}$ then $a - b \neq 0 \Rightarrow \frac{1}{a-b}$ exists

$$\Rightarrow \frac{1}{a-b} \odot \left((a-b) \odot \underline{u}\right) = \frac{1}{(a-b)} \odot \underline{0}$$

By (7) $\left(\frac{a-b}{a-b}\right) \odot \underline{u} = \underline{0} \Rightarrow 1 \odot \underline{u} = \underline{0}$

By (8) $\Rightarrow$ $\underline{u} = \underline{0}$ But we are given that

$\underline{u} \neq 0$ hence $a \neq b$ is not possible

$\Rightarrow$ $\underline{\underline{a = b}}$, as required.

Section 4.3:

(5) $V = R^3$

(a) $W = \left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \right\}$

- $\underline{v_1} = \begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} a_2 \\ b_2 \\ 1 \end{bmatrix}$ in W

$$\underline{v_1} \oplus \underline{v_2} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 2 \end{bmatrix} \text{ not in } W$$

$\Rightarrow W$ not a subspace.

(b) $W = \left\{ \begin{bmatrix} a \\ b \\ a+2b \end{bmatrix} \right\}$

- $\underline{v_1} = \begin{bmatrix} a_1 \\ b_1 \\ a_1 + 2b_1 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} a_2 \\ b_2 \\ a_2 + 2b_2 \end{bmatrix}$ in W

$$\underline{v_1} \oplus \underline{v_2} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ (a_1+a_2) + 2(b_1+b_2) \end{bmatrix} \text{ also in } W$$

- c real no. $c \odot \underline{v_1} = \begin{bmatrix} ca_1 \\ cb_1 \\ ca_1 + 2(cb_1) \end{bmatrix}$ also in W

$\Rightarrow W$ is a subspace.

(c) $W = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \right\}$

- $\underline{v_1} = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix}$ in W $\quad \underline{v_1} \oplus \underline{v_2} = \begin{bmatrix} a_1 + a_2 \\ 0 \\ 0 \end{bmatrix}$ also in W

- c real no. $c \odot \underline{v_1} = \begin{bmatrix} ca_1 \\ 0 \\ 0 \end{bmatrix}$ also in W

$\Rightarrow W$ is a subspace.

(d) $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + 2b - c = 0 \right\}$

- $\underline{v_1} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ in W

i.e. $a_1 + 2b_1 - c_1 = 0$, $a_2 + 2b_2 - c_2 = 0$

$$\underline{v_1} \oplus \underline{v_2} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} : (a_1+a_2) + 2(b_1+b_2) - (c_1+c_2) = (a_1 + 2b_1 - c_1) + (a_2 + 2b_2 - c_2) = 0 + 0 = 0.$$

- c real no. $c \odot \underline{v_1} = \begin{bmatrix} ca_1 \\ cb_1 \\ cc_1 \end{bmatrix}$ $ca_1 + 2cb_1 - cc_1 = c(a_1 + 2b_1 - c_1)$

$= c \cdot 0 = 0.$

also in W

$\Rightarrow W$ is a subspace.

(8) $V = R_4$

(a) $W = \{[0 \ \ 0 \ \ c \ \ d]\}$

- $\underline{v_1} = [0 \ \ 0 \ \ c_1 \ \ d_1]$, $\underline{v_2} = [0 \ \ 0 \ \ c_2 \ \ d_2]$ in W

$\underline{v_1} \oplus \underline{v_2} = [0 \ \ 0 \ \ c_1 + c_2 \ \ d_1 + d_2]$ also in W

- c real no. $c \odot \underline{v_1} = [0 \ \ 0 \ \ cc_1 \ \ cd_1]$ also in W

$\Rightarrow W$ is a subspace

(b) $W = \{[a \ \ b \ \ c \ \ d] : a = 1, \ b = 0, \ a + d = 1\}$

$= \{[1 \ \ 0 \ \ c \ \ 0]\}$

- $\underline{v_1} = [1 \ \ 0 \ \ c_1 \ \ 0]$ $\underline{v_2} = [1 \ \ 0 \ \ c_2 \ \ 0]$ in W

$\underline{v_1} \oplus \underline{v_2} = [2 \ \ 0 \ \ c_1 + c_2 \ \ 0]$ <u>not</u> in W

$\Rightarrow W$ <u>not</u> a subspace

(c) $W = \{[a \ \ b \ \ c \ \ d] : a > 0 \ \ b < 0\}$

<u>Not</u> closed under $\odot$: $c = -1$ $\underline{v} = [1 \ \ -2 \ \ 0 \ \ 0]$

$c \odot \underline{v} = [-1 \ \ 2 \ \ 0 \ \ 0]$ not in W

$\Rightarrow W$ <u>not</u> a subspace.

(9) $V = M_{23}$

(a) $W = \left\{ \begin{bmatrix} a & a+c & c \\ d & 0 & 0 \end{bmatrix} \right\}$

- $\underline{v}_1 = \begin{bmatrix} a_1 & a_1+c_1 & c_1 \\ d_1 & 0 & 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} a_2 & a_2+c_2 & c_2 \\ d_2 & 0 & 0 \end{bmatrix}$ in W

$\underline{v}_1 \oplus \underline{v}_2 = \begin{bmatrix} a_1+a_2 & (a_1+a_2)+(c_1+c_2) & c_1+c_2 \\ d_1+d_2 & 0 & 0 \end{bmatrix}$ also in W

- c real no. $\quad c \odot \underline{v}_1 = \begin{bmatrix} ca_1 & ca_1+cc_1 & cc_1 \\ cd_1 & 0 & 0 \end{bmatrix}$ also in W

$\Rightarrow$ W is a subspace.

(b) $W = \left\{ \begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix} : c > 0 \right\}$

not closed under $\odot$: $-1 \odot \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -3 & -1 \\ -4 & 0 & 0 \end{bmatrix}$ not in W

$\Rightarrow$ W <u>not</u> a subspace

(c) $W = \left\{ \begin{bmatrix} -2c & b & c \\ d & e & 2e+d \end{bmatrix} \right\}$

- $\underline{v}_1 = \begin{bmatrix} -2c_1 & b_1 & c_1 \\ d_1 & e_1 & 2e_1+d_1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -2c_2 & b_2 & c_2 \\ d_2 & e_2 & 2e_2+d_2 \end{bmatrix}$ in W

$\underline{v}_1 \oplus \underline{v}_2 = \begin{bmatrix} -2(c_1+c_2) & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & e_1+e_2 & 2(e_1+e_2)+(d_1+d_2) \end{bmatrix}$ also in W

- c a real no. $\quad c \odot \underline{v}_1 = \begin{bmatrix} -2cc_1 & cb_1 & cc_1 \\ cd_1 & ce_1 & 2ce_1+cd_1 \end{bmatrix}$ also in W

$\Rightarrow$ W is a <u>subspace</u>.

(13) ~~Let~~ $V = M_{22}$ $\quad W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+b+c+d=0 \right\}$

- $\underline{v}_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ in W

$a_1+b_1+c_1+d_1=0$ $\quad a_2+b_2+c_2+d_2=0$

$\underline{v}_1 \oplus \underline{v}_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$ also in W since

$(a_1+a_2)+(b_1+b_2)+(c_1+c_2)+(d_1+d_2)$

$= (a_1+b_1+c_1+d_1) + (a_2+b_2+c_2+d_2)$

$= 0+0 = 0$

- c real no. $\quad c \odot \underline{v}_1 = \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix}$ also in W since

$ca_1 + cb_1 + cc_1 + cd_1$

$= c(a_1+b_1+c_1+d_1) = c \cdot 0$

$= 0$

$\Rightarrow$ W is a subspace.