# Solutions to HW2

## Section 1.4

(22) $A\underline{x} = r\underline{x} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} \Rightarrow \boxed{r=3}$

(29) $AB = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 15 \\ 5 & -4 \end{bmatrix} \Rightarrow (AB)^T = \begin{bmatrix} 11 & 5 \\ 15 & -4 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 15 & -4 \end{bmatrix} \Rightarrow (AB)^T = B^T A^T.$

(31) $(kA)^T(kA) = \begin{bmatrix} -2k & k & -k \end{bmatrix}\begin{bmatrix} -2k \\ k \\ -k \end{bmatrix} = 4k^2 + k^2 + k^2 = 6k^2$

$6k^2 = 1 \Rightarrow \boxed{k = \pm \frac{1}{\sqrt{6}}}$

(38) $\underline{x}_1, \underline{x}_2$ are sol$^n$s to $A\underline{x} = \underline{b} \Rightarrow A\underline{x}_1 = \underline{b}$ and $A\underline{x}_2 = \underline{b}$

we have to show that $A(\underline{x}_1 - \underline{x}_2) = 0$.

$A(\underline{x}_1 - \underline{x}_2) = A\underline{x}_1 - A\underline{x}_2 = \underline{b} - \underline{b} = \underline{0}$ ✓ $\Rightarrow \underline{x}_1 - \underline{x}_2$ is a sol$^n$ to $A\underline{x} = \underline{0}$

## Section 1.5:

(24) (b) $A$ & $B$ are symmetric $\Rightarrow A^T = A$ & $B^T = B$

we have to show $(AB)^T = AB \Leftrightarrow AB = BA$

(i) First assume $(AB)^T = AB$, we will show $AB = BA$

$AB = (AB)^T = B^T A^T = BA$ ✓

(ii) Next assume $AB = BA$, we will show that $(AB)^T = AB$

$(AB)^T = B^T A^T = BA = AB$ ✓

(28) $A^T = -A$ is given.

$$(A^k)^T = \underbrace{(A \dots A)^T}_{k \text{ times}} = \underbrace{(A^T \dots A^T)}_{k \text{ times}} = \underbrace{(-A) \dots (-A)}_{k \text{ times}}$$

$$= (-1)^k A^k \qquad \text{If } k \text{ is odd}, \ (-1)^k = -1$$

$$\Rightarrow (A^k)^T = -A^k \Rightarrow A^k \text{ is skew-symmetric}$$

(30) $A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew-symm}}$ we have $A^T = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 6 & 1 \\ -2 & 2 & 3 \end{bmatrix}$

$$\frac{1}{2}(A + A^T) = \begin{bmatrix} 1 & 7/2 & 3/2 \\ 7/2 & 6 & 3/2 \\ 3/2 & 3/2 & 3 \end{bmatrix}, \quad \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -1/2 & -7/2 \\ 1/2 & 0 & 1/2 \\ 7/2 & -1/2 & 0 \end{bmatrix}$$

(33) (a) $A = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$ Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $AA^{-1} = I_2 \Rightarrow \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a+3c & b+3d \\ 5a+2c & 5b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$a + 3c = 1$ $\quad b + 3d = 0$

$5a + 2c = 0$ $\quad 5b + 2d = 1$

$a = -2/13$ $\quad b = 3/13$

$c = \frac{5}{13}$ $\quad d = -1/13$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} -2/13 & 3/13 \\ 5/13 & -1/13 \end{bmatrix}}$$

(b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $AA^{-1} = I_2 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a+2c & b+2d \\ 2a+c & 2b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$a + 2c = 1$ $\quad b + 2d = 0$

$2a + c = 0$ $\quad 2b + d = 1$

$a = -1/3$ $\quad b = 2/3$

$c = 2/3$ $\quad d = -1/3$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}}$$

(38) $A^2 \underline{x} = \underline{b} \Rightarrow A^{-1}A^2\underline{x} = A^{-1}\underline{b} \Rightarrow A\underline{x} = A^{-1}\underline{b} \Rightarrow A^{-1}A\underline{x} = A^{-1}A^{-1}\underline{b}$

$$\Rightarrow \underline{x} = A^{-1}A^{-1}\underline{b} \Rightarrow \cancel{\underline{x} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}} \quad \underline{x} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \underline{x} = \begin{bmatrix} 9 & 0 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 \\ -6 \end{bmatrix} \Rightarrow \boxed{\underline{x} = \begin{bmatrix} -9 \\ -6 \end{bmatrix}}$$

Section 1.6

(8) $f(\underline{x}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \underline{x}$

(9) $\underline{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ We have to find a $\underline{x}$ such that $f(\underline{x}) = \underline{w}$

Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$

$\Rightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \underline{w} \Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_2 = 1 \\ x_2 = -1 \\ x_1 + x_2 = 2 \end{matrix} \Rightarrow \boxed{\begin{matrix} x_2 = -1 \\ x_1 = 3 \end{matrix}}$

$\Rightarrow \underline{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is in the range of $f$.

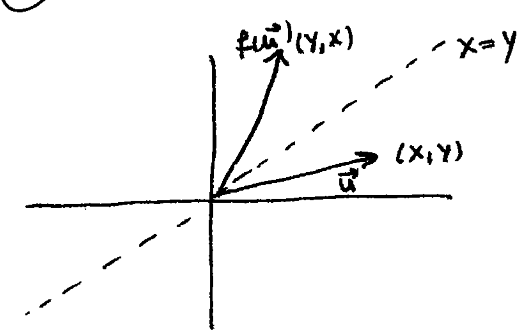
(13) $\underline{w} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ $\quad f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \underline{w} \Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

$\Rightarrow$
$x_1 + 2x_2 = 1 \longrightarrow x_1 = -7$
$x_2 = 4 \longrightarrow x_2 = 4$
$x_1 + x_2 = 2 \longrightarrow x_1 = -2$

2 diff values of $x_1 \Rightarrow$ no sol$^n$.

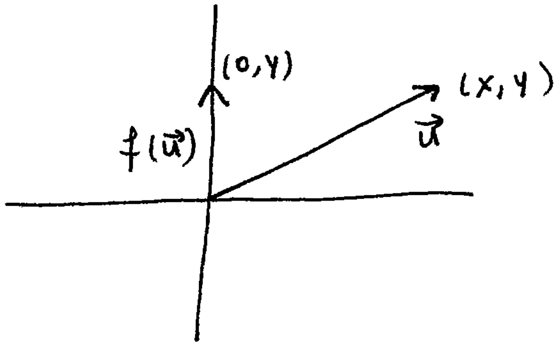
$\Rightarrow \underline{w} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ not in the range of $f$.

(16) (a) $f(\underline{u}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}$ $\quad$ Let $\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ then $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$




$f$ is reflection about the line $x = y$.

(17) (b) $f(\vec{u}) = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \underline{u}$ Let $\underline{u} = \begin{bmatrix}x\\ y\end{bmatrix}$ $f\left(\begin{bmatrix}x\\ y\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ y\end{bmatrix}$

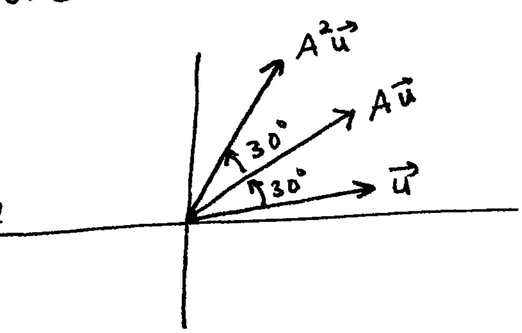



$f$ is projection onto the $y$-axis.

(19) $f(\underline{u}) = \underbrace{\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}}_{A} \underline{u}$ with $\phi = 30^\circ$

So, $f$ is a counterclockwise rotation by $30^\circ$.

(a) $T_1(\underline{u}) = A^2 \underline{u}$




$\Rightarrow T_1$ is counterclockwise rotation by $60^\circ$

(b) $T_2(\underline{u}) = A^{-1}\underline{u}$ Since $A^{-1}A\underline{u} = I\underline{u} = \underline{u}$, $A^{-1}$ does the exact opposite of $A$ $\Rightarrow$ $A^{-1}$ is clockwise rotation by $30^\circ$.

(c) $T_3(\underline{u}) = A^k \underline{u}$. We have seen that each application of $A$ moves ~~us~~ $\underline{u}$ counterclockwise by $30^\circ$. If we want $A^k\underline{u} = \underline{u}$, we have to rotate all of $360^\circ$ to return to $\underline{u}$. If each rotation ~~step~~ is $30^\circ$ then we require a minimum of 12 ~~steps~~ rotations to get $360^\circ$. Hence min value of $k$ such that $A^k\underline{u} = \underline{u}$ is

$\boxed{k = 12}$