MATH 3333

Midterm I

September 20, 2007

Name :

I.D. no. :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.
- Show as much work as possible. Answers without explanation will not receive any credit.
- Best of Luck.

i) **(20 Points)** Find all the values of a such that the following system has

a) no solution,

b) unique solution and

c) infinitely many solutions.

$$\begin{aligned} x + y &= 4 \\ x + (a^2 - 15)y &= a \end{aligned}$$

The augmented matrix corresponding to the above linear system is given by

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & a^2 - 15 & a \end{array}\right)$$

Applying the row operation $-\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2$ we get

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a^2 - 16 & a - 4 \end{array}\right)$$

Now there are 3 cases.

a) $\mathbf{a = 4}$: In this case the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

and the linear system of equations is $x + y = 4$. Hence there are **infinitely many solutions** given by $x = r, y = 4 - r$, where r is any real number.

b) $\mathbf{a = -4}$: In this case the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & -8 \end{array}\right)$$

and the linear system of equations is $x + y = 4, 0 = -8$. There are **no solutions** since the second condition can never be satisfied.

c) $\mathbf{a \neq \pm 4}$: In this case apply the row operation $\frac{1}{a^2-16}\mathbf{r}_2 \to \mathbf{r}_2$ to get

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & \frac{1}{a+4} \end{array}\right)$$

This gives us the linear system

$$x + y = 4, \qquad y = \frac{1}{a+4}$$

which has the **unique solution** $x = 4 - \frac{1}{a+4}, y = \frac{1}{a+4}$.

ii) (20 **Points**)

a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. Find A^{-1}, if it exists. Record the row operations you perform, using the notation for elementary row operations.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \qquad -\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \qquad -\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right) \qquad -\mathbf{r}_3 \to \mathbf{r}_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) \qquad -2\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2, -\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) \qquad -\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

Hence we have

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$$

b) Find the solution to the following linear system $A\mathbf{x} = \mathbf{b}$ where A is as in part (a) above and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ by A^{-1} we get

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

iii) **(20 Points)** Are the following two system of linear equations equivalent ? Explain your answer.

$$\begin{aligned} 4x + 3y &= 1 \\ -x + 6y &= -7 \end{aligned} \qquad\qquad \begin{aligned} 5x - 2y &= 7 \\ -2x + y &= -3 \end{aligned}$$

Two linear systems are said to be equivalent if they have the exact same set of solutions.

Consider the first system $4x + 3y = 1, -x + 6y = -7$. Solving for x in the second equation gives us $x = 6y + 7$. Substituting this in the first equation we get $4(6y + 7) + 3y = 1$. Solving for y we get $y = -1$ and hence $x = 1$. So the unique solution to the first system is given by $x = 1, y = -1$.

Consider the second system $5x - 2y = 7, -2x + y = -3$. Solving for y in the second equation gives us $y = 2x - 3$. Substituting this in the first equation we get $5x - 2(2x - 3) = 7$. Solving for x we get $x = 1$ and hence $y = -1$. So the unique solution to the second system is given by $x = 1, y = -1$.

Since both the systems have the unique solution $x = 1, y = -1$ they are equivalent systems.

iv) (20 **Points**)

a) Give a geometric description of the matrix transformation given by $T_1(\mathbf{u}) = A_1\mathbf{u}$ where $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Observe that $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$. We know that the matrix transformation corresponding to $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ is a clockwise rotation by an angle θ. Hence T_1 is a clockwise rotation by an angle $\pi/2$.

b) The matrix transformation $T_2(\mathbf{u}) = A_2\mathbf{u}$ where $A_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is a clockwise rotation. Determine the angle of rotation.

Setting $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ we get $\cos(\theta) = 1/\sqrt{2}$ and $\sin(\theta) = 1/\sqrt{2}$. This implies that $\theta = \pi/4$. Hence T_2 is a clockwise rotation by an angle $\pi/4$.

c) Give a geometric description of the matrix transformation given by $T_3(\mathbf{u}) = A_3\mathbf{u}$, where $A_3 = A_2A_1$ with A_1, A_2 as above.

The action of T_3 is the same as first applying T_1 and then applying T_2. This means that to any vector $\mathbf{u}$ we first apply a clockwise rotation of an angle $\pi/2$ and then we apply a clockwise rotation of an angle $\pi/4$. Hence T_3 is a clockwise rotation by an angle $3\pi/4$.

v) **(20 Points)** Determine whether the following statements are true or false. Explain your answers.

a) If A and B are $n \times n$ matrices then $((A^{-1}B^T)^{-1})^T = A^T B^{-1}$.
TRUE

$$((A^{-1}B^T)^{-1})^T = ((B^T)^{-1}A)^T = A^T((B^T)^{-1})^T = A^T B^{-1}$$

b) If A and B are 2×2 matrices then $(A+B)^2 = A^2 + 2AB + B^2$.
FALSE

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$$

if $AB \neq BA$.

c) If A and B are 2×2 invertible matrices, then $A + B$ is also an invertible matrix.
FALSE
Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then both A and B are invertible. But $A + B$ is the zero matrix which is not invertible.

d) The matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
TRUE
Two matrices are equivalent if one can be obtained from the other by elementary row or column operations. Apply the row operation $-3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2$ to A to get $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$. Now apply the column operation $-4\mathbf{c}_1 + \mathbf{c}_2 \to \mathbf{c}_2$ to get $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.