Math 3113, Quiz III Solution

Use Laplace Transform to solve the initial value problem

$$x'' - x' - 2x = e^{3t}, \qquad x(0) = 0, x'(0) = 0.$$

Take the Laplace transform of the differential equation to get

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 2\mathcal{L}\{x\} = \mathcal{L}\{e^{3t}\}.$$

Using the formulae for the Laplace transform of derivatives we get

$$(s^2X(s) - sx(0) - x'(0)) - (sX(s) - x(0)) - 2X(s) = \frac{1}{s-3}$$

where $X(s) = \mathcal{L}\{x\}$. Using the initial values and solving for $X(s)$ we get

$$X(s) = \frac{1}{(s-3)(s^2-s-2)} = \frac{1}{(s-3)(s-2)(s+1)}$$

Now we have to use partial fractions to write

$$\frac{1}{(s-3)(s-2)(s+1)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s+1}$$

Equalizing the denominator on both sides we get

$$1 = A(s-2)(s+1) + B(s-3)(s+1) + C(s-3)(s-2)$$

Substitute $s = 3$ to get $1 = 4A \Rightarrow A = 1/4$.

Substitute $s = 2$ to get $1 = -3B \Rightarrow B = -1/3$.

Substitute $s = -1$ to get $1 = 12C \Rightarrow C = 1/12$.

Hence we have

$$X(s) = \frac{1}{4}\left(\frac{1}{s-3}\right) - \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{1}{12}\left(\frac{1}{s+1}\right)$$

Now using $\mathcal{L}\{e^{at}\} = 1/(s-a)$ or equivalently $\mathcal{L}^{-1}\{1/(s-a)\} = e^{at}$ we get

$$\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{4}e^{3t} - \frac{1}{3}e^{2t} + \frac{1}{12}e^{-t}.$$