On the standard L-function for $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$ and algebraicity of symmetric fourth L-values for GL_2

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Abstract

We prove an explicit integral representation – involving the pullback of a suitable Siegel Eisenstein series – for the twisted standard L-function associated to a holomorphic vector-valued Siegel cusp form of degree n and arbitrary level. In contrast to all previously proved pullback formulas in this situation, our formula involves only scalar-valued functions despite being applicable to L-functions of vector-valued Siegel cusp forms. The key new ingredient in our method is a novel choice of local vectors at the archimedean place which allows us to exactly compute the archimedean local integral.

By specializing our integral representation to the case $n = 2$ we are able to prove a reciprocity law – predicted by Deligne's conjecture – for the critical special values of the twisted standard L-function for vector-valued Siegel cusp forms of degree 2 and arbitrary level. This arithmetic application generalizes previously proved critical-value results for the full level case. By specializing further to the case of Siegel cusp forms obtained via the Ramakrishnan–Shahidi lift, we obtain a reciprocity law for the critical special values of the symmetric fourth L-function of a classical newform.

Contents

1 Introduction

1.1 Critical L-values

The critical values of L-functions attached to cohomological cuspidal automorphic representations are objects of deep arithmetic significance. In particular, a famous conjecture of Deligne [12] predicts that these values are algebraic numbers up to multiplication by suitable periods[1]. One can go even further and ask for the prime ideal factorizations of these algebraic numbers (once the period is suitably normalized). This goes beyond Deligne's conjecture and is related to the Bloch-Kato conjectures.

    The first results on the algebraicity of critical L-values of automorphic representations were obtained by Shimura [42, 43, 44], who considered automorphic representations of GL_2 (and twists and Rankin-Selberg products of these). For higher rank groups, initial steps were taken by Harris [20] and Sturm [51] in 1981, who considered automorphic representations of $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$ whose finite part is unramified and whose infinity type is a holomorphic discrete series with scalar minimal K-type. Since then, there has been considerable work in this area and algebraicity results for L-values of automorphic representations on various algebraic groups have been proved; the general problem, however, remains very far from being resolved.

    In this paper we revisit the case of the standard L-function on $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$. As mentioned earlier, this is the first case outside GL_2 that was successfully tackled, with the (independent) results of Harris and Sturm back in 1981. However, the automorphic representations considered therein were very special and corresponded, from the classical point of view, to scalar-valued Siegel cusp forms of full level. Subsequent works on the critical L-values of Siegel cusp forms by Böcherer [5], Mizumoto [31], Shimura [49], Böcherer–Schmidt [8], Kozima [29], Bouganis [10] and others strengthened and extended these results in several directions. Nonetheless, a proof of algebraicity of critical L-values for holomorphic forms on $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$ in full generality has not yet been achieved, *even for $n = 2$.*

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[1]However, it is often a non-trivial problem to relate the automorphic periods to Deligne's motivic periods.

We prove the following result for $n = 2$, which applies to representations whose finite part is arbitrary and whose archimedean part can be (almost) any holomorphic discrete series.[2]

1.1 Theorem. *Let $k_1 \geq k_2 \geq 3$, $k_1 \equiv k_2$ (mod 2) be integers. For each cuspidal automorphic representation π on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with π_∞ isomorphic to the holomorphic discrete series representation with highest weight (k_1, k_2), there exists a real number $C(\pi)$ with the following properties.*

   i) $C(\pi) = C(\pi')$ whenever π and π' are nearly equivalent.

  ii) $C(\pi) = C(\pi \otimes (\psi \circ \mu))$ for any Dirichlet character ψ. Here $\mu : \mathrm{GSp}_4 \to \mathrm{GL}_1$ denotes the standard multiplier map.

 iii) Let χ be a Dirichlet character satisfying $\chi(-1) = (-1)^{k_2}$ and r be an integer such that $1 \leq r \leq k_2 - 2$, $r \equiv k_2$ (mod 2). Furthermore, if $\chi^2 = 1$, we assume that $r \neq 1$. Then, for any finite subset S of places of $\mathbb{Q}$ including the archimedean place, and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\sigma \left(\frac{L^S(r, \pi \boxtimes \chi, \varrho_5)}{(2\pi i)^{3r} G(\chi)^3 C(\pi)} \right) = \frac{L^S(r, {}^\sigma\pi \boxtimes {}^\sigma\chi, \varrho_5)}{(2\pi i)^{3r} G({}^\sigma\chi)^3 C({}^\sigma\pi)}, \tag{1}$$

and therefore in particular

$$\frac{L^S(r, \pi \boxtimes \chi, \varrho_5)}{(2\pi i)^{3r} G(\chi)^3 C(\pi)} \in \mathbb{Q}(\pi, \chi).$$

Above, $G(\chi)$ denotes the Gauss sum, $L^S(s, \pi \boxtimes \chi, \varrho_5)$ denotes the degree 5 L-function (after omitting the local factors in S) associated to the representation $\pi \boxtimes \chi$ of $\mathrm{GSp}_4 \times \mathrm{GL}_1$, and $\mathbb{Q}(\pi, \chi)$ denotes the (CM) field generated by χ and the field of rationality for π.

Before going further, we make a few remarks pertaining to the statement above.

1.2 Remark. *Note that we can take $S = \{\infty\}$. In particular, this gives us an algebraicity result for the full finite part of the global L-function. This is an important point because some previous works on this topic [8, 46, 49] either omit the bad local L-factors, impose some extra conditions on them, or define these bad factors in an ad-hoc manner. We take a different approach, whereby we first prove our theorem for the unramified part of the global L-function, and then put back the finitely many well-defined bad L-factors using the local Langlands correspondence. The functorial transfer from Sp_4 to GL_5 together with our detailed knowledge of lifts (see [2, 40]) implies that the local and global L-factors appearing in Theorem 1.1 do not have poles; the $\mathrm{Aut}(\mathbb{C})$ equivariance of the bad local factors likewise follows directly from work of Clozel [11].*

1.3 Remark. *Our proofs show that $C(\pi) = \pi^{2k_1} \langle F, F \rangle$ where F equals a certain **nearly holomorphic** modular form of **scalar** weight k_1. Alternatively one can take $C(\pi) = \pi^{k_1+k_2} \langle F_0, F_0 \rangle$ where F_0 equals a certain holomorphic **vector-valued** modular form of weight $\det^{k_2} \mathrm{sym}^{k_1-k_2}$.*

1.4 Remark. *Observe that the L-function $L(s, \pi \boxtimes \chi, \varrho_5)$ considered above is not equal to the L-function $L(s, \pi \otimes (\chi \circ \mu), \varrho_5)$. In fact, $L(s, \pi \boxtimes \chi, \varrho_5)$ cannot be obtained as the L-function of an automorphic representation on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, unless χ is trivial.*

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[2]We omit the ones that do not contribute to cohomology, namely those with $k_1 \not\equiv k_2$ (mod 2).

Classically, Theorem 1.1 applies to vector-valued holomorphic Siegel cusp forms of weight $\det^{k_2} \operatorname{sym}^{k_1-k_2}$ with respect to an arbitrary congruence subgroup of $\operatorname{Sp}_4(\mathbb{Q})$. The only previously known result for critical L-values of holomorphic Siegel cusp forms in the vector-valued case $(k_1 > k_2)$ is due to Kozima [29]. Kozima's result only applies to full-level Siegel cusp forms, omits some low-weight cases, and also only deals with the case $\chi = 1$. In contrast, our theorem, which relies on an adelic machinery to separate out the difficulties place by place, is more general, especially in that it applies to arbitrary congruence subgroups.

The main achievement of this paper, however, is *not* Theorem 1.1 above but an explicit integral representation (Theorem 1.6 below) for the standard L-function $L(s, \pi \boxtimes \chi, \varrho_{2n+1})$ on $\operatorname{GSp}_{2n} \times \operatorname{GL}_1$. This integral representation applies to any $\pi \boxtimes \chi$ such that π_∞ is a holomorphic discrete series representation of highest weight $(k_1, k_2, \ldots, k_n)$ with all k_i of the same parity, and $\chi_\infty = \operatorname{sgn}^{k_1}$ (there is no restriction on the finite part of π). While Theorem 1.6 is formally similar to the well-known pullback formula (or doubling method) mechanism due to Garrett, Shimura, Piatetski-Shapiro–Rallis and others, what distinguishes it from previous works is that (despite the generality of our setup) all our constants are completely explicit. Previously, such an explicit pullback formula had been proved only for very special archimedean types. The key innovation that makes Theorem 1.6 possible is a certain novel choice of local vectors which allows us to exactly compute all local zeta integrals appearing in the pullback formula, including at the ramified and archimedean places. These points are explained in more detail later in the introduction.

Note that Theorem 1.6 applies to any n; we restrict to $n = 2$ only in Section 7 of this paper, where we prove Theorem 1.1. At the end of that section, in Theorem 7.9, we present an application to critical special values of the symmetric fourth L-function of elliptic newforms twisted by odd Dirichlet characters. We state a mild variant of Theorem 7.9 below.

1.5 Theorem. *Let $k \geq 2$ be even. For each cuspidal, non-dihedral, automorphic representation η on $\operatorname{PGL}_2(\mathbb{A}_{\mathbb{Q}})$ with η_∞ isomorphic to the holomorphic discrete series representation of lowest weight k, there exists a real number $C(\eta)$ with the following properties.*

i) *$C(\eta) = C(\eta \otimes \psi)$ for any quadratic Dirichlet character ψ.*

ii) *Let χ be an odd Dirichlet character and r be an odd integer such that $1 \leq r \leq k - 1$. Furthermore, if $\chi^2 = 1$, we assume that $r \neq 1$. Then, for any finite subset S of places of $\mathbb{Q}$ that includes the archimedean place, and any $\sigma \in \operatorname{Aut}(\mathbb{C})$, we have*

$$\sigma\left(\frac{L^S(r, \chi \otimes \operatorname{sym}^4 \eta)}{(2\pi i)^{3r} G(\chi)^3 C(\eta)}\right) = \frac{L^S(r, {}^\sigma\chi \otimes \operatorname{sym}^4({}^\sigma\eta))}{(2\pi i)^{3r} G({}^\sigma\chi)^3 C({}^\sigma\eta)}. \tag{2}$$

Our proof of Theorem 1.5 relies on a result of Ramakrishnan and Shahidi [37] which states that given an elliptic cuspidal newform of even weight and trivial nebentypus, there exists a holomorphic vector-valued Siegel cusp form of genus 2 such that the degree 5 standard L-function of the Siegel modular form is equal to the symmetric fourth L-function of the elliptic newform. This allows us to derive Theorem 1.5 from Theorem 1.1. One of the reasons to prove Theorem 1.1 with no restrictions on the non-archimedean components is the incomplete information regarding the congruence subgroup associated to the Siegel modular form in [37].

1.2 Integral representations for $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$ and the pullback formula

The first integral representation for the standard (degree $2n + 1$) L-function for automorphic representations of $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$ was discovered by Andrianov and Kalinin [1] in 1978. The integral representation of Andrianov–Kalinin applied to holomorphic scalar-valued Siegel cusp forms of even degree n with respect to $\Gamma_0(N)$ type congruence subgroups, and involved a certain theta series. They used their integral representation to prove results about the meromorphic continuation of the L-function; they also derived the functional equation when 8 divides n. The results of Harris [20] and Sturm [51] mentioned earlier also relied on this integral representation.

A remarkable new integral representation, commonly referred to as the pullback formula, was discovered in the early 1980s by Garrett [17] and involved pullbacks of Eisenstein series. Roughly speaking, the pullback formula in the simplest form says that

$$\int\limits_{\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}_n} F(-\overline{Z_1})E_k\left(\begin{bmatrix} Z_1 & \\ & Z_2 \end{bmatrix}, s\right)dZ_1 \approx L(2s + k - n, \pi, \varrho_{2n+1})F(Z_2), \tag{3}$$

where F is a Siegel cusp form of degree n and full level that is an eigenform for all the Hecke operators, $E_k(Z, s)$ is an Eisenstein series of degree $2n$ and full level (that is, a holomorphic Siegel modular form of weight k when $s = 0$), $L(s, \pi, \varrho_{2n+1})$ denotes the degree $2n+1$ L-function for π, and the symbol $\approx$ indicates that we are ignoring some unimportant factors. The pullback formula was applied by Böcherer [5] to prove various results about the functional equation and algebraicity of critical L-values that went well beyond what had been possible by the Andrianov–Kalinin formula. Subsequently, Shimura generalized the pullback formula to a wide variety of contexts (including other groups). We refer to Shimura's books [48, 49] for further details.

In the last two decades, the pullback formula has been used to prove a host of results related to the arithmetic and analytic properties of L-functions associated to Siegel cusp forms. However, most of these results involve significant restrictions. To get the most general results possible, it is necessary to extend (3) to a) incorporate *characters*, b) include Siegel cusp forms with respect to arbitrary *congruence subgroups*, and c) cover the case of *vector-valued* Siegel cusp forms. The first two of these objectives have to a large extent been achieved for scalar-valued Siegel cusp forms. However, the situation with vector-valued forms is quite different. Following the important work of Böcherer–Satoh–Yamazaki [7], there have been a few results about vector-valued forms by Takei [54], Takayanagi [52, 53], Kozima [29, 30], and others. Unfortunately, all these works are valid only for full level Siegel cusp forms and involve strong restrictions on the archimedean type.[3] To get the most general pullback formula for vector-valued Siegel cusp form, it seems necessary to use the adelic approach. Most of the results about vector-valued forms described above use certain differential operators involving tensor products of vector-valued functions [7, 25] and it seems unclear if these can be incorporated into an adelic treatment.

On the other hand, Piatetski-Shapiro and Rallis [34, 19] discovered a very general identity (the doubling method) on classical groups. When the group is Sp_{2n}, this is essentially a generalized, adelic version of the pullback formula described above.[4] However, Piatetski-Shapiro and Rallis computed the relevant local integrals only at the unramified places (where they chose the

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[3]There is also a survey talk by Böcherer (Hakuba, 2012) on the pullback formula for more general vector-valued forms, but it still restricts to full level and does not incorporate characters.

[4]However, at least in the original formulation, the doubling method did not incorporate characters.

vectors to be the unramified vector). Thus, to get an explicit integral representation from the basic identity of Piatetski-Shapiro and Rallis for more general automorphic representations, it is necessary to make specific choices of vectors at the ramified and archimedean primes such that one can exactly evaluate all the associated local integrals. So far, this has been carried out in very few situations. It seems worth mentioning here the recent work of Zheng Liu (Ph.D. thesis in Columbia) who uses the doubling method for vector-valued Siegel modular forms and shows the non-vanishing of the archimedean integral (without computing it explicitly) and also suggests some algebraicity results (without proving a reciprocity law).

In this paper, we begin by reproving the basic identity of Piatetski-Shapiro and Rallis in a somewhat different manner that is more convenient for our purposes. Let π be a cuspidal automorphic representation of $\mathrm{GSp}_{2n}(\mathbb{A})$ and χ a Hecke character. Our central object of investigation is the global integral

$$Z(s; f, \phi)(g) = \int\limits_{\mathrm{Sp}_{2n}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})} E^\chi((h, g), s, f) \phi(h) \, dh. \tag{4}$$

Here, $E^\chi(-, s, f)$ is an Eisenstein series on $\mathrm{GSp}_{4n}(\mathbb{A})$ associated to a choice of section $f \in I(\chi, s)$, the pair (h, g) represents an element of $\mathrm{GSp}_{4n}(\mathbb{A})$ corresponding to the diagonal embedding of $\mathrm{GSp}_{2n}(\mathbb{A}) \times \mathrm{GSp}_{2n}(\mathbb{A})$, and ϕ is an automorphic form in the space of π. The integral (4) represents a meromorphic function of s on all of $\mathbb{C}$ due to the analytic properties of the Eisenstein series. As we will prove, away from any poles, the function $g \mapsto Z(s; f, \phi)(g)$ is again an *automorphic form in the space of* π.

Next, assume that $\phi = \otimes_v \phi_v$ and $f = \otimes f_v$ are factorizable. We define, for each place v, *local zeta integrals*

$$Z_v(s; f_v, \phi_v) := \int\limits_{\mathrm{Sp}_{2n}(F_v)} f_v(Q_n \cdot (h, 1), s) \pi_v(h) \phi_v \, dh, \tag{5}$$

where Q_n is a certain explicit matrix in $\mathrm{GSp}_{4n}(\mathbb{A})$. It turns out (see Proposition 3.2) that $Z_v(s; f_v, \phi_v)$ converges to an element in the space of π_v for real part of s large enough.

Our "Basic Identity" (Theorem 3.6) asserts that the automorphic form $Z(s; f, \phi)$ corresponds to the pure tensor $\otimes_v Z_v(s; f_v, \phi_v)$. Now assume that, for all v, the vectors ϕ_v and the sections f_v can be chosen in such a way that $Z_v(s, f_v, \phi_v) = c_v(s) \phi_v$ for an explicitly computable function $c_v(s)$. Then we obtain a *global integral representation*

$$Z(s; f, \phi)(g) = \Big(\prod_v c_v(s) \Big) \phi. \tag{6}$$

In this way the Euler product $\prod_v c_v(s)$, convergent for $\mathrm{Re}(s)$ large enough, inherits analytic properties from the left hand side of (6). If this Euler product, up to finitely many factors and up to "known" functions, represents an interesting L-function, one can thus derive various properties of said L-function.

Our main local task is therefore to choose the vectors ϕ_v and the sections f_v such that $Z_v(s, f_v, \phi_v) = c_v(s) \phi_v$ for an explicitly computable function $c_v(s)$. For a "good" non-archimedean place v we will make the obvious unramified choices. The *unramified calculation*, Proposi-

tion 4.1, then shows that

$$c_v(s) = \frac{L((2n+1)s+1/2, \pi_v \boxtimes \chi_v, \varrho_{2n+1})}{L((2n+1)(s+1/2), \chi_v) \prod\limits_{i=1}^{n} L((2n+1)(2s+1) - 2i, \chi_v^2)}. \tag{7}$$

If v is non-archimedean and some of the data is *ramified*, it is possible to choose ϕ_v and f_v such that $c_v(s)$ is a non-zero constant function; see Proposition 4.3. The idea is to choose ϕ_v to be invariant under a small enough principal congruence subgroup, and make the support of f_v small enough. This idea was inspired by [32].

1.3 The choice of archimedean vectors and our main formula

We now explain our choice of ϕ_v and f_v at a real place v, which represents one of the main new ideas of this paper. We only treat the case of π_v being a holomorphic discrete series representation, since this is sufficient for our application to Siegel modular forms. Assume first that π_v has scalar minimal K-type of weight k, i.e., π_v occurs as the archimedean component attached to a classical Siegel cusp form of weight k. Then it is natural to take for ϕ_v a vector of weight k (spanning the minimal K-type of π_v), and for f_v a vector of weight k in $I(\chi_v, s)$. Both ϕ_v and f_v are unique up to scalars. The local integral in this case is calculated in Proposition 5.5 and turns out to be an exponential times a rational function. The calculation is made possible by the fact that we have a simple formula for the matrix coefficient of ϕ_v; see (65).

Now assume that π_v is a holomorphic discrete series representation with more general minimal K-type $\rho_{(k_1,\ldots,k_n)}$, where $k_1 \geq \ldots \geq k_n > n$; see Sect. 5 for notation. In this case we will *not* work with the minimal K-type, but with the scalar K-type $\rho_{(k,\ldots,k)}$, where $k := k_1$. We will show in Lemma 5.3 that $\rho_{(k,\ldots,k)}$ occurs in π_v with multiplicity one. Let ϕ_v be the essentially unique vector spanning this K-type, and let f_v again be the vector of weight k in $I(\chi_v, s)$. The function $c_v(s)$ in this case is again an exponential times a rational function; see Proposition 5.8. We note that our calculation for general minimal K-type uses the result of the scalar case, so the scalar case cannot be subsumed in the more general case. One difficulty in the general case is that we do not know a formula for the matrix coefficient of ϕ_v, or in fact *any* matrix coefficient. Instead, we use a completely different method, realizing ϕ_v as a vector in an induced representation.

Our archimedean calculations require two different integral formulas, which are both expressions for the Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$. The first formula, used in the scalar minimal K-type case, is with respect to the KAK decomposition; see (27). The second formula, used in the general case, is with respect to the Iwasawa decomposition; see (85). We will always normalize the Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$ in the "classical" way, characterized by the property (12). It is necessary for us to determine the precise constants relating the KAK measure and the Iwasawa measure to the classical normalization. This will be carried out in Appendix A.

Finally, combining the archimedean and the non-archimedean calculations, we obtain an explicit formula for the right side of (6), which is our aforementioned pullback formula for L-functions on $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$. This is Theorem 6.1 in the main body of the paper; below we state a rough version of this theorem.

1.6 Theorem. *Let* $\pi \cong \otimes \pi_p$ *be an irreducible cuspidal automorphic representation of* $\mathrm{GSp}_{2n}(\mathbb{A}_{\mathbb{Q}})$ *such that* π_∞ *is a holomorphic discrete series representation of highest weight*

$(k_1, k_2, \ldots, k_n)$, where $k = k_1 \geq \ldots \geq k_n > n$ and all k_i have the same parity. Let $\chi = \otimes \chi_p$ be a character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ such that $\chi_\infty = \mathrm{sgn}^k$. Then we have the identity

$$Z(s, f, \phi)(g) = \frac{L^S((2n+1)s + 1/2, \pi \boxtimes \chi, \varrho_{2n+1})}{L^S((2n+1)(s+1/2), \chi) \prod_{j=1}^n L^S((2n+1)(2s+1) - 2j, \chi^2)}$$
$$\times \, i^{nk} \, \pi^{n(n+1)/2} c((2n+1)s - 1/2)\phi(g),$$

where $c(z)$ is an explicit function that takes non-zero rational values for integers z with $0 \leq z \leq k_n - n$ and where $L^S(\ldots)$ denotes the global L-functions without the local factors corresponding to a certain (explicit) finite set of places S.

The above formula can be reformulated in a classical language, which takes a similar form to (3) and involves functions $F(Z)$ and $E^\chi_{k,N}(Z, s)$ that correspond to ϕ and $E^\chi(-, s, f)$ respectively. In fact F and $E^\chi_{k,N}(-, s)$ are (scalar-valued) smooth modular forms of weight k (and degrees n and $2n$ respectively) with respect to suitable congruence subgroups. We refer the reader to Theorem 6.3 for the full classical statement. In all previously proved classical pullback formulas [7, 52, 53] for $L(s, \pi \boxtimes \chi)$ with π_∞ a general discrete series representation, the analogues of F and $E^\chi_{k,N}(-, s)$ were vector-valued objects; in contrast, our formula involves only scalar-valued functions. This is a key point of the present work.

We hope that our pullback formula will be useful for arithmetic applications beyond what we pursue in this paper. A particularly fruitful direction might be towards congruence primes and the Bloch-Kato conjecture, extending work of Agarwal, Berger, Brown, Klosin, and others.

1.4 Nearly holomorphic modular forms and arithmeticity

To obtain results about the algebraicity of critical L-values, we delve deeper into the arithmetic nature of the two smooth modular forms given above. The general arithmetic principle here is that whenever a smooth modular/automorphic form is holomorphic, or close to being holomorphic, it is likely to have useful arithmetic properties. In this case, if $0 \leq m \leq \frac{k-n-1}{2}$ is an integer, then Shimura has proved that $E^\chi_{k,N}(Z, -m)$ is a *nearly holomorphic* Siegel modular form (of degree $2n$) with nice arithmetic properties.

The next step is to prove that the inner product of $F(Z)$ and $E^\chi_{k,N}(Z, -m)$ is $\mathrm{Aut}(\mathbb{C})$ equivariant. It is here that we are forced to assume $n = 2$. In this case, our recent investigation of lowest weight modules [35] of $\mathrm{Sp}_4(\mathbb{R})$ and in particular the "structure theorem" proved therein allows us to define an $\mathrm{Aut}(\mathbb{C})$ equivariant isotypic projection map from the space of *all* nearly holomorphic modular forms to the subspace of cusp forms corresponding to a particular infinity-type. Once this is known, Theorem 1.1 follows by a standard linear algebra argument going back at least to Garrett [18].

It is worth contrasting our approach here with previously proved results on the critical L-values of holomorphic vector-valued Siegel cusp forms such as the result of Kozima [29] mentioned earlier. In Kozima's work, the modular forms involved in the integral representation are vector-valued and the cusp form holomorphic; ours involves two scalar-valued modular forms that are not holomorphic. Our approach allows us to incorporate everything smoothly into an adelic setup and exactly evaluate the archimedean integral. But the price we pay is that the arithmeticity of the automorphic forms is not automatic (as we cannot immediately appeal to the arithmetic geometry inherent in holomorphic modular forms). In particular, this is the reason we are forced

to restrict ourselves to $n = 2$ in the final section of this paper, where we prove Theorem 1.1. We expect that an analogue of the structure theorem for nearly holomorphic forms proved in [35] for $n = 2$ should continue to hold for general n. This would immediately lead to an extension of Theorem 1.1 for any n.

1.5   Acknowledgements

2   Preliminaries

2.1   Basic notations and definitions

Let F be an algebraic number field and $\mathbb{A}$ the ring of adeles of F. For a positive integer n let G_{2n} be the algebraic F-group GSp_{2n}, whose F-points are given by

$$G_{2n}(F) = \{g \in \mathrm{GL}_{2n}(F) \ : \ {}^t g J_n g = \mu_n(g) J_n, \ \mu_n(g) \in F^\times\}, \qquad J_n = \left[\begin{smallmatrix} & I_n \\ -I_n & \end{smallmatrix} \right].$$

The symplectic group Sp_{2n} consists of those elements $g \in G_{2n}$ for which the multiplier $\mu_n(g)$ is 1. Let P_{2n} be the Siegel parabolic subgroup of G_{2n}, consisting of matrices whose lower left $n \times n$-block is zero. Let $\delta_{P_{2n}}$ be the modulus character of $P_{2n}(\mathbb{A})$. It is given by

$$\delta_{P_{2n}}\left(\left[\begin{smallmatrix} A & X \\ & v\,{}^tA^{-1} \end{smallmatrix} \right] \right) = |v^{-\frac{n(n+1)}{2}} \det(A)^{n+1}|, \qquad \text{where } A \in \mathrm{GL}_n(\mathbb{A}), \ v \in \mathrm{GL}_1(\mathbb{A}), \tag{8}$$

and $|\cdot|$ denotes the global absolute value, normalized in the standard way.

Fix the following embedding of $H_{2a,2b} := \{(g, g') \in G_{2a} \times G_{2b} : \mu_a(g) = \mu_b(g')\}$ in G_{2a+2b}:

$$H_{2a,2b} \ni \left(\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix} \right], \left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix} \right] \right) \longmapsto \begin{bmatrix} A_1 & & -B_1 & \\ & A_2 & & B_2 \\ -C_1 & & D_1 & \\ & C_2 & & D_2 \end{bmatrix} \in \mathrm{GSp}_{2a+2b}. \tag{9}$$

We will also let $H_{2a,2b}$ denote its image in G_{2a+2b}.

Let G be any reductive algebraic group defined over F. For a place v of F let (π_v, V_v) be an *admissible representation* of $G(F_v)$. If v is non-archimedean, then this means that every vector in V_v is smooth, and that for every open-compact subgroup Γ of $G(F_v)$ the space of fixed vectors V_v^Γ is finite-dimensional. If v is archimedean, then it means that V_v is an admissible $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ is the Lie algebra of $G(F_v)$ and K is a maximal compact subgroup of $G(F_v)$. We say that π_v is *unitary* if there exists a $G(F_v)$-invariant (non-archimedean case) resp. $\mathfrak{g}$-invariant (archimedean case) hermitian inner product on V_v. In this case, and assuming that π_v is irreducible, we can complete V_v to a unitary Hilbert space representation $\bar{V}_v$, which is unique up to unitary isomorphism. We can recover V_v as the subspace of $\bar{V}_v$ consisting of smooth (non-archimedean case) resp. K-finite (archimedean case) vectors.

We define automorphic representations as in [9]. In particular, when we say "automorphic representation of $G(\mathbb{A})$", we understand that at the archimedean places we do not have actions of $G(F_v)$, but of the corresponding $(\mathfrak{g}, K)$-modules. All automorphic representations are assumed to be irreducible. Cuspidal automorphic representations are assumed to be unitary. Any such

representation π is isomorphic to a restricted tensor product $\otimes \pi_v$, where π_v is an irreducible, admissible, unitary representation of $G(F_v)$.

For a place v of F, let $\sigma, \chi_1, \cdots, \chi_n$ be characters of $F_v^\times$. We denote by $\chi_1 \times \cdots \times \chi_n \rtimes \sigma$ the representation of $G_{2n}(F_v)$ parabolically induced from the character

$$\begin{bmatrix} a_1 & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & & \vdots \\ & & a_n & * & \cdots & * \\ & & & a_0 a_1^{-1} & & \\ & & & \vdots & \ddots & \\ & & & * & \cdots & a_0 a_n^{-1} \end{bmatrix} \longmapsto \chi_1(a_1) \cdot \ldots \cdot \chi_n(a_n) \sigma(a_0) \tag{10}$$

of the standard Borel subgroup of $G_{2n}(F_v)$. Restricting all functions in the standard model of this representation to $\mathrm{Sp}_{2n}(F_v)$, we obtain a Borel-induced representation of $\mathrm{Sp}_{2n}(F_v)$ which is denoted by $\chi_1 \times \ldots \times \chi_n \rtimes 1$.

We also define parabolic induction from $P_{2n}(F_v)$. Let χ and σ be characters of $F_v^\times$. Then $\chi \rtimes \sigma$ is the representation of $G_{2n}(F_v)$ parabolically induced from the character

$$\begin{bmatrix} A & * \\ & u\, {}^t A^{-1} \end{bmatrix} \longmapsto \sigma(u) \chi(\det(A)) \tag{11}$$

of $P_{2n}(F_v)$. The center of $G_{2n}(F_v)$ acts on $\chi \rtimes \sigma$ via the character $\chi^n \sigma^2$. Restricting the functions in the standard model of $\chi \rtimes \sigma$ to $\mathrm{Sp}_{2n}(F_v)$, we obtain the Siegel-induced representation of $\mathrm{Sp}_{2n}(F_v)$ denoted by $\chi \rtimes 1$.

We fix a Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$, as follows. Let $\mathbb{H}_n$ be the Siegel upper half space of degree n, consisting of all complex, symmetric $n \times n$ matrices $X + iY$ with X, Y real and Y positive definite. The group $\mathrm{Sp}_{2n}(\mathbb{R})$ acts transitively on $\mathbb{H}_n$ in the usual way. The stabilizer of the point $I := i 1_n \in \mathbb{H}_n$ is the maximal compact subgroup $K = \mathrm{Sp}_{2n}(\mathbb{R}) \cap \mathrm{O}(2n) \cong U(n)$. We transfer the classical $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant measure on $\mathbb{H}_n$ to a left-invariant measure on $\mathrm{Sp}_{2n}(\mathbb{R})/K$. We also fix a Haar measure on K so that K has volume 1. The measures on $\mathrm{Sp}_{2n}(\mathbb{R})/K$ and K define a unique Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$. Let F be a measurable function on $\mathrm{Sp}_{2n}(\mathbb{R})$ that is right K-invariant. Let f be the corresponding function on $\mathbb{H}_n$, i.e., $F(g) = f(gI)$. Then, by these definitions,

$$\int_{\mathrm{Sp}_{2n}(\mathbb{R})} F(h)\, dh = \int_{\mathbb{H}_n} f(Z) \det(Y)^{-(n+1)}\, dX\, dY. \tag{12}$$

We shall always use the Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$ characterized by the property (12).

Haar measures on $\mathrm{Sp}_{2n}(F)$, where F is a non-archimedean field with ring of integers $\mathfrak{o}$, will be fixed by requiring that the open-compact subgroup $\mathrm{Sp}_{2n}(\mathfrak{o})$ has volume 1. The Haar measure on an adelic group $\mathrm{Sp}_{2n}(\mathbb{A})$ will always be taken to be the product measure of all the local measures. (At the complex places, the precise choice of measure will be irrelevant for this work, so one can make an arbitrary choice.)

2.2 Some coset decompositions

For $0 \leq r \leq n$, let $\alpha_r \in \mathrm{Sp}_{4n}(\mathbb{Q})$ be the matrix

$$\alpha_r = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & \widetilde{I}_r & I_n & 0 \\ \widetilde{I}_r & 0 & 0 & I_n \end{bmatrix}, \tag{13}$$

where the $n \times n$ matrix $\widetilde{I_r}$ is given by $\widetilde{I_r} = \begin{bmatrix} 0_{n-r} & 0 \\ 0 & I_r \end{bmatrix}$.

2.1 Proposition. *We have the double coset decomposition*

$$G_{4n}(F) = \bigsqcup_{r=0}^{n} P_{4n}(F)\alpha_r H_{2n,2n}(F).$$

Proof. This follows from Shimura [48, Prop. 2.4]. ∎

For our purposes, it is nicer to work with the coset representatives $Q_r := \alpha_r \cdot (I_{4n-2r}, J_r)$ where $J_r = \begin{bmatrix} & I_r \\ -I_r & \end{bmatrix}$. It is not hard to see that (I_{4n-2r}, J_r) is actually an element of $H_{2n,2n}$, so that

$$P_{4n}(F)\alpha_r H_{2n,2n}(F) = P_{4n}(F)Q_r H_{2n,2n}(F).$$

One can write down the matrix Q_r explicitly as follows,

$$Q_r = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I'_{n-r} & 0 & \widetilde{I_r} \\ 0 & 0 & I_n & \widetilde{I_r} \\ \widetilde{I_r} & -\widetilde{I_r} & 0 & I'_{n-r} \end{bmatrix}, \tag{14}$$

where $I'_{n-r} = I_n - \widetilde{I_r} = \begin{bmatrix} I_{n-r} & 0 \\ 0 & 0 \end{bmatrix}$.

For $0 \le r \le n$, let $P_{2n,r}$ be the maximal parabolic subgroup (proper if $r \ne n$) of G_{2n} consisting of matrices whose lower-left $(n+r) \times (n-r)$ block is zero. Its Levi component is isomorphic to $\mathrm{GL}_{n-r} \times G_r$. Note that $P_{2n,0} = P_{2n}$ and $P_{2n,n} = G_{2n}$. Let $N_{2n,r}$ denote the unipotent radical of $P_{2n,r}$.

The next lemma expresses the reason why $\{Q_r\}$ is more convenient than $\{\alpha_r\}$ for the double coset representatives. Let $d : P_{4n} \to \mathrm{GL}_1$ be the homomorphism defined by

$$d(p) = v^{-n} \det(A)$$

for $p = \begin{bmatrix} A & \\ & v \, {}^t\!A^{-1} \end{bmatrix} n'$ with $v \in \mathrm{GL}_1, A \in \mathrm{GL}_{2n}$ and $n' \in N_{4n,0}$. Note that $\delta_{P_{4n}}(p) = |d(p)|^{2n+1}$ for $p \in P_{4n}(\mathbb{A})$.

2.2 Lemma.   *i) Let $0 \le r < n$ and suppose that*

$$p_1 = \begin{bmatrix} g_1 & & & \\ & I_r & & \\ & & {}^t g_1^{-1} & \\ & & & I_r \end{bmatrix} \cdot n_1, \quad p_2 = \begin{bmatrix} g_2 & & & \\ & I_r & & \\ & & {}^t g_2^{-1} & \\ & & & I_r \end{bmatrix} \cdot n_2, \quad g_1, g_2 \in \mathrm{GL}_{n-r}, \quad n_1, n_2 \in N_{2n,r}$$

are certain elements of $P_{2n,r}$. Then the matrix $X = Q_r(p_1, p_2)Q_r^{-1}$ lies in P_{4n} and satisfies $d(X) = \det(g_1)\det(g_2)$.

ii) Let $0 \le r < n$. Let $x_1 \in \mathrm{Sp}_{2r}$ and $x = (1, x_1) \in \mathrm{Sp}_{2n}$. Then the matrix $X = Q_r(x, x)Q_r^{-1}$ lies in P_{4n} and satisfies $d(X) = 1$.

iii) Let $g \in G_{2n}$. Then the matrix $X = Q_n(g, g)Q_n^{-1}$ lies in P_{4n} and satisfies $d(X) = 1$.

Proof. This follows by direct verification. ∎

Next we provide a set of coset representatives for $P_{4n}(F)\backslash P_{4n}(F)Q_r H_{2n,2n}(F)$.

2.3 Proposition. *For each $0 \leq r \leq n$, we have the coset decomposition*

$$P_{4n}(F)Q_rH_{2n,2n}(F) = \bigsqcup_{\substack{x=(1,x_1)\in\mathrm{Sp}_{2n}(F) \\ x_1\in\mathrm{Sp}_{2r}(F)}} \bigsqcup_{\substack{y\in P_{2n,r}(F)\backslash G_{2n}(F) \\ z\in P_{2n,r}(F)\backslash G_{2n}(F)}} P_{4n}(F)\cdot Q_r \cdot (xy,z),$$

where we choose the representatives y, z to be in $\mathrm{Sp}_{2n}(F)$.

Proof. This follows from Shimura [48, Prop. 2.7]. Note that the choice of representatives y and z is irrelevant by i) and ii) of Lemma 2.2. ∎

3   Eisenstein series and zeta integrals

3.1   Degenerate principal series representations

Let χ be a character of $F^\times\backslash\mathbb{A}^\times$. We define a character on $P_{4n}(\mathbb{A})$, also denoted by χ, by $\chi(p) = \chi(d(p))$. For a complex number s, let

$$I(\chi, s) = \mathrm{Ind}_{P_{4n}(\mathbb{A})}^{G_{4n}(\mathbb{A})}(\chi\delta_{P_{4n}}^s). \tag{15}$$

Thus, $f(\,\cdot\,, s) \in I(\chi, s)$ is a smooth complex-valued function satisfying

$$f(pg, s) = \chi(p)\delta_{P_{4n}}(p)^{s+\frac{1}{2}}f(g, s) \tag{16}$$

for all $p \in P_{4n}(\mathbb{A})$ and $g \in G_{4n}(\mathbb{A})$. Note that these functions are invariant under the center of $G_{4n}(\mathbb{A})$. Let $I_v(\chi_v, s)$ be the analogously defined local representation. Using the notation introduced in Sect. 2.1, we have

$$I_v(\chi_v, s) = \chi_v|\cdot|_v^{(2n+1)s} \rtimes \chi_v^{-n}|\cdot|_v^{-n(2n+1)s}. \tag{17}$$

We have $I(\chi, s) \cong \otimes I_v(\chi_v, s)$ in a natural way. Given $f \in I(\chi, s)$, it follows from iii) of Lemma 2.2 that, for all $h \in \mathrm{Sp}_{2n}(\mathbb{A})$,

$$f(Q_n \cdot (gh, g), s) = f(Q_n \cdot (h, 1), s) \qquad \text{for } g \in G_{2n}(\mathbb{A}). \tag{18}$$

We will mostly use this observation in the following form. Let $f_v \in I_v(\chi_v, s)$ and K a maximal compact subgroup of $\mathrm{Sp}_{2n}(F_v)$. Then

$$f_v(Q_n \cdot (k_1hk_2, 1), s) = f_v(Q_n \cdot (h, 1)(k_2, k_1^{-1}), s) \qquad \text{for } k_1, k_2 \in K \tag{19}$$

and all $h \in \mathrm{Sp}_{2n}(F_v)$.

  In preparation for the next result, and for the unramified calculation in Sect. 4, we recall some facts concerning the unramified Hecke algebra at a non-archimedean place v of F. We fix a uniformizer ϖ of F_v. Let $K = \mathrm{Sp}_{2n}(\mathfrak{o}_v)$. Recall the Cartan decomposition

$$\mathrm{Sp}_{2n}(F) = \bigsqcup_{\substack{e_1,\cdots,e_n\in\mathbb{Z} \\ 0\leq e_1\leq\cdots\leq e_n}} K_{e_1,\cdots,e_n}, \tag{20}$$

where

$$K_{e_1,\cdots,e_n} = K \operatorname{diag}(\varpi^{e_1}, \cdots, \varpi^{e_n}, \varpi^{-e_1}, \cdots, \varpi^{-e_n})K. \tag{21}$$

Consider the spherical Hecke algebra $\mathcal{H}_n$ consisting of left and right K-invariant functions on $\operatorname{Sp}_{2n}(F)$. The structure of this Hecke algebra is described by the Satake isomorphism

$$\mathcal{S}: \mathcal{H}_n \longrightarrow \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^W, \tag{22}$$

where the superscript W indicates polynomials that are invariant under the action of the Weyl group of Sp_{2n}. Let $T_{e_1,\ldots,e_n}$ be the characteristic function of the set $K_{e_1,\ldots,e_n}$ defined in (21). Then $T_{e_1,\ldots,e_n}$ is an element of the Hecke algebra $\mathcal{H}_n$. The values $\mathcal{S}(T_{e_1,\ldots,e_n})$ are encoded in the *rationality theorem*

$$\sum_{\substack{e_1,\ldots,e_n\in\mathbb{Z}\\ 0\le e_1\le\ldots\le e_n}} \mathcal{S}(T_{e_1,\ldots,e_n})Y^{e_1+\ldots+e_n} = \frac{1-Y}{1-q^nY}\prod_{i=1}^{n}\frac{1-q^{2i}Y^2}{(1-X_iq^nY)(1-X_i^{-1}q^nY)}. \tag{23}$$

This identity of formal power series is the main result of [6].

Let $\chi_1,\ldots,\chi_n$ be unramified characters of $F_v^\times$. Let π be the unramified constituent of $\chi_1 \times \ldots \times \chi_n \rtimes 1$. The numbers $\alpha_i = \chi_i(\varpi)$, $i = 1,\ldots,n$, are called the *Satake parameters* of π. Let v_0 be a spherical vector in π. It is unique up to scalars. Hence, if we act on v_0 by an element T of $\mathcal{H}_n$, we obtain a multiple of v_0. This multiple is given by evaluating $\mathcal{S}T$ at the Satake parameters, i.e.,

$$\int_{\operatorname{Sp}_{2n}(F)} T(x)\pi(x)v_0\,dx = (\mathcal{S}T)(\alpha_1,\ldots,\alpha_n)v_0. \tag{24}$$

Now assume that v is a real place. Let $K = \operatorname{Sp}_{2n}(\mathbb{R}) \cap \operatorname{O}(2n) \cong U(n)$ be the standard maximal compact subgroup of $\operatorname{Sp}_{2n}(\mathbb{R})$. Let $\mathfrak{g}$ be the (uncomplexified) Lie algebra of $\operatorname{Sp}_{2n}(\mathbb{R})$, and let $\mathfrak{a}$ be the subalgebra consisting of diagonal matrices. Let Σ be the set of restricted roots with respect to $\mathfrak{a}$. If e_i is the linear map sending $\operatorname{diag}(a_1,\ldots,a_n,-a_1,\ldots,-a_n)$ to a_i, then Σ consists of all $\pm(e_i - e_j)$ for $1 \le i < j \le n$ and $\pm(e_i + e_j)$ for $1 \le i \le j \le n$. As a positive system we choose

$$\Sigma^+ = \{e_j - e_i : 1 \le i < j \le n\} \cup \{-e_i - e_j : 1 \le i \le j \le n\} \tag{25}$$

(what is more often called a *negative* system). Then the positive Weyl chamber is

$$\mathfrak{a}^+ = \{\operatorname{diag}(a_1,\ldots,a_n,-a_1,\ldots,-a_n) : a_1 < \ldots < a_n < 0\}. \tag{26}$$

By Proposition 5.28 of [27], or Theorem 5.8 in Sect. I.5 of [23], we have the integration formula

$$\int_{\operatorname{Sp}_{2n}(\mathbb{R})} \phi(h)\,dh = \alpha_n \int_K \int_{\mathfrak{a}^+} \int_K \left(\prod_{\lambda\in\Sigma^+} \sinh(\lambda(H))\right)\phi(k_1\exp(H)k_2)\,dk_1\,dH\,dk_2, \tag{27}$$

which we will use for continuous, non-negative functions ϕ on $\operatorname{Sp}_{2n}(\mathbb{R})$. The measure dH in (27) is the Lebesgue measure on $\mathfrak{a}^+ \subset \mathbb{R}^n$. The positive constant α_n relates the Haar measure given by the integration on the right hand side to the Haar measure dh on $\operatorname{Sp}_{2n}(\mathbb{R})$ we fixed once and for all by (12). We will calculate α_n explicitly in Appendix A.1.

3.2 Local zeta integrals

3.1 Lemma. *i) Let v be any place of F and let $f_v \in I_v(\chi_v, s)$. Then, for $\mathrm{Re}(s)$ large enough, the function on $\mathrm{Sp}_{2n}(F_v)$ defined by $h \mapsto f_v(Q_n \cdot (h, 1), s)$ is in $L^1(\mathrm{Sp}_{2n}(F_v))$.*

ii) Let $f \in I(\chi, s)$. Then, for $\mathrm{Re}(s)$ large enough, the function on $\mathrm{Sp}_{2n}(\mathbb{A})$ defined by $h \mapsto f(Q_n \cdot (h, 1), s)$ is in $L^1(\mathrm{Sp}_{2n}(\mathbb{A}))$.

Proof. Since ii) follows from i) by definition of the adelic measure, we only have to prove the local statement. To ease notation, we will omit all subindices v. Define a function $f'(g, s)$ by

$$f'(g, s) = \int_K \int_K |f(g(k_2, k_1^{-1}), s)| \, dk_1 \, dk_2, \qquad g \in G_{4n}(F). \tag{28}$$

From (16), we see that

$$f'(pg, s) = |\chi(p)\delta_{P_{4n}}(p)^{s+\frac{1}{2}}| \, f'(g, s) \tag{29}$$

for all $p \in P_{4n}(F)$ and $g \in G_{4n}(F)$. Equation (19) implies that

$$\int_{\mathrm{Sp}_{2n}(F)} |f(Q_n \cdot (h, 1), s)| \, dh = \int_{\mathrm{Sp}_{2n}(F)} |f'(Q_n \cdot (h, 1), s)| \, dh. \tag{30}$$

From now on we will assume that v is a non-archimedean place; the proofs for the real and complex places are similar. It follows from (20) that

$$\int_{\mathrm{Sp}_{2n}(F)} |f'(Q_n \cdot (h, 1), s)| \, dh = \sum_{\substack{e_1, \cdots, e_n \in \mathbb{Z} \\ 0 \le e_1 \le \cdots \le e_n}} \int_{K_{e_1, \cdots, e_n}} |f'(Q_n \cdot (h, 1), s)| \, dh$$

$$= \sum_{\substack{e_1, \cdots, e_n \in \mathbb{Z} \\ 0 \le e_1 \le \cdots \le e_n}} \mathrm{vol}(K_{e_1, \cdots, e_n}) \left| f'(Q_n \cdot (\mathrm{diag}(\varpi^{e_1}, \cdots, \varpi^{e_n}, \varpi^{-e_1}, \cdots, \varpi^{-e_n}), 1), s) \right|. \tag{31}$$

From (29), we find

$$f'(Q_n \cdot (\mathrm{diag}(\varpi^{e_1}, \cdots, \varpi^{e_n}, \varpi^{-e_1}, \cdots, \varpi^{-e_n}), 1), s)$$

$$= \left(|\chi(\varpi)| q^{-(2n+1)(\mathrm{Re}(s)+1/2)} \right)^{e_1 + \cdots + e_n} f'(\begin{bmatrix} I_n & & & \\ & I_n & & \\ & A & I_n & \\ A & & & I_n \end{bmatrix} \begin{bmatrix} I_n & & & \\ & I_n & & I_n \\ & & I_n & \\ & -I_n & & \end{bmatrix}, s) \tag{32}$$

with $A = \mathrm{diag}(\varpi^{e_1}, \ldots, \varpi^{e_n})$. By smoothness, the term $f'(\ldots)$ in the second line of (32) takes only finitely many values, and can therefore be estimated by a constant C independent of $e_1, \ldots, e_n$. Thus

$$\int_{\mathrm{Sp}_{2n}(F)} |f'(Q_n \cdot (h, 1), s)| \, dh \le C \sum_{\substack{e_1, \cdots, e_n \in \mathbb{Z} \\ 0 \le e_1 \le \cdots \le e_n}} \mathrm{vol}(K_{e_1, \cdots, e_n}) \, c(s)^{e_1 + \cdots + e_n}, \tag{33}$$

where $c(s) = |\chi(\varpi)| q^{-(2n+1)(\mathrm{Re}(s)+1/2)}$. Consider the trivial representation $1_{\mathrm{Sp}_{2n}(F)}$ of $\mathrm{Sp}_{2n}(F)$. Since it is a subrepresentation of $| \cdot |^{-n} \times \ldots \times | \cdot |^{-1}$, its Satake parameters are $\alpha_i = q^i$ for

$i = 1, \ldots, n$. Let v_0 be a spanning vector of $1_{\mathrm{Sp}_{2n}(F)}$. Then $T_{e_1,\ldots,e_n} v_0 = \mathrm{vol}(K_{e_1,\cdots,e_n}) v_0$. By (24) it follows that $\mathrm{vol}(K_{e_1,\cdots,e_n}) = \mathcal{S}(T_{e_1,\ldots,e_n})(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i = q^i$. Substituting $\alpha_i = q^i$ into (23), we get

$$\sum_{0 \le e_1 \le \ldots \le e_n} \mathrm{vol}(K_{e_1,\cdots,e_n}) Y^{e_1 + \ldots + e_n} = \frac{1-Y}{1-q^n Y} \prod_{i=1}^{n} \frac{1 + q^i Y}{1 - q^{n+i} Y}, \tag{34}$$

an identity of formal power series in Y. We see that (33) is convergent if $c(s)$ is sufficiently small, i.e., if $\mathrm{Re}(s)$ is sufficiently large. ∎

3.2 Proposition. *Let χ be a character of $F^\times \backslash \mathbb{A}^\times$.*

 i) *For a place v of F let $f_v \in I_v(\chi_v, s)$. Let (π_v, V_v) be an admissible, unitary representation of $\mathrm{Sp}_{2n}(F_v)$. If $\mathrm{Re}(s)$ is sufficiently large, then the integral*

$$Z_v(s; f_v, w_v) := \int\limits_{\mathrm{Sp}_{2n}(F_v)} f_v(Q_n \cdot (h, 1), s) \pi_v(h) w_v \, dh \tag{35}$$

 converges absolutely to an element of $\bar{V}_v$, for any w_v in the Hilbert space completion $\bar{V}_v$. If w_v is in V_v, then so is $Z_v(s; f_v, w_v)$.

 ii) *Let $\pi \cong \otimes \pi_v$ be a cuspidal, automorphic representation of $G_{2n}(\mathbb{A})$. Let V be the space of automorphic forms on which π acts. If $\mathrm{Re}(s)$ is sufficiently large, then the function*

$$g \longmapsto \int\limits_{\mathrm{Sp}_{2n}(\mathbb{A})} f(Q_n \cdot (h, 1), s) \phi(gh) \, dh \tag{36}$$

 is an element of V, for every $\phi \in V$. If $f = \otimes f_v$ with $f_v \in I_v(\chi_v, s)$, and if ϕ corresponds to the pure tensor $\otimes w_v$, then the function (36) corresponds to the pure tensor $\otimes Z_v(s; f_v, w_v)$.

Proof. i) The absolute convergence follows form Lemma 3.1 i). The second assertion can be verified in a straightforward way using (19).

ii) Lemma 3.1 ii) implies that the integral

$$\int\limits_{\mathrm{Sp}_{2n}(\mathbb{A})} f(Q_n \cdot (h, 1), s)(R(h)\phi) \, dh, \tag{37}$$

where R denotes right translation, converges absolutely to an element in the Hilbert space completion $\bar{V}$ of V. With the same argument as in the local case we see that this element has the required smoothness properties that make it an automorphic form, thus putting it into V. Evaluating at g, we obtain the first assertion. The second assertion follows by applying a unitary isomorphism $\pi \cong \otimes \pi_v$ to (37). ∎

3.3 The basic identity

Let $I(\chi, s)$ be as in (15), and let $f(\cdot, s)$ be a section whose restriction to the standard maximal compact subgroup of $G_{4n}(\mathbb{A})$ is independent of s. Consider the Eisenstein series on $G_{4n}(\mathbb{A})$ which, for $\operatorname{Re}(s) > \frac{1}{2}$, is given by the absolutely convergent series

$$E(g, s, f) = \sum_{\gamma \in P_{4n}(F) \backslash G_{4n}(F)} f(\gamma g, s), \tag{38}$$

and defined by analytic continuation outside this region. Let π be a cuspidal automorphic representation of $G_{2n}(\mathbb{A})$. Let V_π be the space of cuspidal automorphic forms realizing π. For any automorphic form ϕ in V_π and any $s \in \mathbb{C}$ define a function $Z(s; f, \phi)$ on $G_{2n}(F) \backslash G_{2n}(\mathbb{A})$ by

$$Z(s; f, \phi)(g) = \int\limits_{\mathrm{Sp}_{2n}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})} E((h, g), s, f) \phi(h) \, dh. \tag{39}$$

3.3 Remark. *Note that* $g \cdot \mathrm{Sp}_{2n}(\mathbb{A}) = \{h \in G_{2n}(\mathbb{A}) : \mu_n(h) = \mu_n(g)\}$.

3.4 Remark. $E(g, s, f)$ *is slowly increasing away from its poles and* ϕ *is rapidly decreasing. So* $Z(s; f, \phi)(g)$ *converges absolutely for* $s \in \mathbb{C}$ *away from the poles of the Eisenstein series and defines an automorphic form on* G_{2n}. *We will see soon that* $Z(s; f, \phi)$ *in fact belongs to* V_π.

Using Propositions 2.1 and 2.3, we conclude that $Z(s; f, \phi)(g) = \sum_{r=0}^{n} Z_r(s; f, \phi)(g)$, where

$$Z_r(s; f, \phi)(g) = \int\limits_{\substack{\mathrm{Sp}_{2n}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})}} \sum_{\substack{x=(1,x_1) \in \mathrm{Sp}_{2n}(F) \\ x_1 \in \mathrm{Sp}_{2r}(F)}} \sum_{\substack{y \in P_{2n,r}(F) \backslash G_{2n}(F) \\ z \in P_{2n,r}(F) \backslash G_{2n}(F)}} f(Q_r \cdot (xyh, zg), s) \phi(h) \, dh.$$

3.5 Proposition. *Let* $0 \le r < n$. *Then* $Z_r(s; f, \phi)(g) = 0$.

Proof. Let $N_{2n,r}$ be the unipotent radical of the maximal parabolic subgroup $P_{2n,r}$. Let $P'_{2n,r} = P_{2n,r} \cap \mathrm{Sp}_{2n}$. Then

$$Z_r(s; f, \phi)(g) = \int\limits_{\substack{\mathrm{Sp}_{2n}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})}} \sum_{\substack{x=(1,x_1) \in \mathrm{Sp}_{2n}(F) \\ x_1 \in \mathrm{Sp}_{2r}(F)}} \sum_{\substack{y \in P'_{2n,r}(F) \backslash \mathrm{Sp}_{2n}(F) \\ z \in P_{2n,r}(F) \backslash G_{2n}(F)}} f(Q_r \cdot (xyh, zg), s) \phi(h) \, dh$$

$$= \int\limits_{\substack{P'_{2n,r}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})}} \sum_{\substack{x=(1,x_1) \in \mathrm{Sp}_{2n}(F) \\ x_1 \in \mathrm{Sp}_{2r}(F)}} \sum_{\substack{z \in P_{2n,r}(F) \backslash G_{2n}(F)}} f(Q_r \cdot (xh, zg), s) \phi(h) \, dh$$

$$= \int\limits_{\substack{N_{2n,r}(\mathbb{A}) P'_{2n,r}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})}} \int\limits_{\substack{N_{2n,r}(F) \backslash N_{2n,r}(\mathbb{A})}} \sum_{x,z} f(Q_r \cdot (xnh, zg), s) \phi(nh) \, dn \, dh.$$

Since the element x normalizes $N_{2n,r}(\mathbb{A})$ and $N_{2n,r}(F)$, we can commute x and n. Then n can be omitted by i) of Lemma 2.2. Hence

$$Z_r(s; f, \phi)(g) = \int\limits_{\substack{N_{2n,r}(\mathbb{A}) P'_{2n,r}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})}} \int\limits_{\substack{N_{2n,r}(F) \backslash N_{2n,r}(\mathbb{A})}} \left(\sum_{x,z} f(Q_r \cdot (xh, zg), s) \right) \phi(nh) \, dn \, dh,$$

and the cuspidality of ϕ implies that this is zero.   ∎

For the following theorem, which is the main result of this section, recall the local integrals defined in (35).

3.6 Theorem. (Basic identity) *Let $\phi \in V_\pi$ be a cusp form which corresponds to a pure tensor $\otimes_v \phi_v$ via the isomorphism $\pi \cong \otimes \pi_v$. Assume that $f \in I(\chi, s)$ factors as $\otimes f_v$ with $f_v \in I(\chi_v, s)$. Let the function $Z(s; f, \phi)$ on $G_{2n}(F) \backslash G_{2n}(\mathbb{A})$ be defined as in (39). Then $Z(s; f, \phi)$ also belongs to V_π and corresponds to the pure tensor $\otimes_v Z_v(s; f_v, \phi_v)$.*

Proof. By Proposition 3.5, we have that $Z(s; f, \phi)(g) = Z_n(s; f, \phi)(g)$. But

$$
\begin{aligned}
Z_n(s; f, \phi)(g) &= \int\limits_{\mathrm{Sp}_{2n}(F) \backslash g \cdot \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{x \in \mathrm{Sp}_{2n}(F)} f(Q_n \cdot (xh, g), s)\phi(h) \, dh \\
&= \int\limits_{g \cdot \mathrm{Sp}_{2n}(\mathbb{A})} f(Q_n \cdot (h, g), s)\phi(h) \, dh \\
&= \int\limits_{\mathrm{Sp}_{2n}(\mathbb{A})} f(Q_n \cdot (h, 1), s)\phi(gh) \, dh,
\end{aligned}
$$

where the last step follows from (18). The theorem now follows from Proposition 3.2.   ∎

Our goal will be to choose, at all places, the vectors ϕ_v and the sections f_v in such a way that $Z_v(s, f_v, \phi_v) = c_v(s)\phi_v$ for an explicitly computable function $c_v(s)$.

4   The local integral at finite places

In this section we define suitable local sections and calculate the local integrals (35) for all finite places v. We will drop the subscript v throughout. Hence, let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be its ring of integers, ϖ a uniformizer, and $\mathfrak{p} = \varpi \mathfrak{o}$ the maximal ideal.

4.1   Unramified places

We begin with the unramified case. Let χ be an unramified character of $F^\times$, and let π be a spherical representation of $G_{2n}(F)$. Let $f \in I(\chi, s)$ be the normalized unramified vector, i.e., $f : G_{4n} \to \mathbb{C}$ is given by

$$
f\left(\begin{bmatrix} A & {}^* \\ & u\,{}^t A^{-1} \end{bmatrix} k\right) = \chi(u^{-n} \det(A)) \big| u^{-n} \det(A) \big|^{(2n+1)(s+1/2)} \tag{40}
$$

for $A \in \mathrm{GL}_{2n}(F)$, $u \in F^\times$ and $k \in G_{4n}(\mathfrak{o})$. Let v_0 be a spherical vector in π. We wish to calculate the local integral

$$
Z(s; f, v_0) = \int\limits_{\mathrm{Sp}_{2n}(F)} f(Q_n \cdot (h, 1), s)\pi(h)v_0 \, dh. \tag{41}
$$

Let $\sigma, \chi_1, \cdots, \chi_n$ be unramified characters of $F^\times$ such that π is the unique spherical constituent of $\chi_1 \times \cdots \times \chi_n \rtimes \sigma$. Let $\alpha_i = \chi_i(\varpi)$, $i = 1, \ldots, n$, and $\alpha_0 = \sigma(\varpi)$ be the Satake parameters of π. The dual group of $\mathrm{GSp}_{2n} \times \mathrm{GL}_1$ is $\mathrm{GSpin}_{2n+1}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$. There is a natural map $\varrho_{2n+1} : \mathrm{GSpin}_{2n+1}(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C}) \to \mathrm{GL}_{2n+1}(\mathbb{C})$. Consequently we get a tensor product map from $\mathrm{GSpin}_{2n+1}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ into $\mathrm{GL}_{2n+1}(\mathbb{C})$ which we denote also by ϱ_{2n+1}. The L-function $L(s, \pi \boxtimes \chi, \varrho_{2n+1})$ is then given as follows,

$$L(s, \pi \boxtimes \chi, \varrho_{2n+1}) = \frac{1}{1 - \chi(\varpi)q^{-s}} \prod_{i=1}^{n} \frac{1}{(1 - \chi(\varpi)\alpha_i q^{-s})(1 - \chi(\varpi)\alpha_i^{-1}q^{-s})}. \tag{42}$$

We also define $L(s, \chi) = (1 - \chi(\varpi)q^{-s})^{-1}$, as usual.

4.1 Proposition. *Using the above notations and hypotheses, the local integral* (41) *is given by*

$$Z(s; f, v_0) = \frac{L((2n+1)s + 1/2, \pi \boxtimes \chi, \varrho_{2n+1})}{L((2n+1)(s+1/2), \chi) \prod\limits_{i=1}^{n} L((2n+1)(2s+1) - 2i, \chi^2)} v_0$$

for real part of s large enough.

Proof. The calculation is similar to that in the proof of Lemma 3.1. By (20), (19) and (32),

$$Z(s; f, v_0) = \sum_{\substack{e_1, \cdots, e_n \in \mathbb{Z} \\ 0 \le e_1 \le \cdots \le e_n}} f(Q_n \cdot \mathrm{diag}(\varpi^{e_1}, \cdots, \varpi^{e_n}, \varpi^{-e_1}, \cdots, \varpi^{-e_n}), 1), s) \int_{K_{e_1, \cdots, e_n}} \pi(h)v_0 \, dh$$

$$= \sum_{\substack{e_1, \cdots, e_n \in \mathbb{Z} \\ 0 \le e_1 \le \cdots \le e_n}} \left(\chi(\varpi)q^{-(2n+1)(s+1/2)} \right)^{e_1 + \cdots + e_n} T_{e_1, \ldots, e_n} v_0,$$

where $T_{e_1, \ldots, e_n} v_0$ is the Hecke operator introduced in Sect. 3.2. The restriction of π to $\mathrm{Sp}_{2n}(F)$ equals the spherical constituent of $\chi_1 \times \ldots \times \chi_n \rtimes 1$, which has Satake parameters $\alpha_i = \chi_i(\varpi)$. The assertion now follows from (24) and (23). ∎

4.2   Ramified places

Now we deal with the ramified places. For a non-negative integer m, let $\Gamma_{2n}(\mathfrak{p}^m) = \{g \in \mathrm{Sp}_{2n}(\mathfrak{o}) : g \equiv I_{2n} \bmod \mathfrak{p}^m\}$.

4.2 Lemma. *Let m be a positive integer. Let $p \in P_{4n}(F)$ and $h \in \mathrm{Sp}_{2n}(F)$ be such that $Q_n^{-1}p \, Q_n(h, 1) \in \Gamma_{4n}(\mathfrak{p}^m)$. Then $h \in \Gamma_{2n}(\mathfrak{p}^m)$ and $p \in P_{4n}(F) \cap \Gamma_{4n}(\mathfrak{p}^m)$.*

Proof. Since Q_n normalizes $\Gamma_{4n}(\mathfrak{p}^m)$, the hypothesis is equivalent to $p \, Q_n(h, 1)Q_n^{-1} \in \Gamma_{4n}(\mathfrak{p}^m)$. Write $p = \left[\begin{smallmatrix} * & * \\ & g \end{smallmatrix}\right]$ with $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in \mathrm{GL}_{2n}(F)$ and $h = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & g_1 & g_2 \\ & & g_3 & g_4 \end{bmatrix} \begin{bmatrix} A & & -B & \\ & I_n & & \\ -C & I_n-D & D & \\ A-I_n & B & -B & I_n \end{bmatrix} \in \Gamma_{4n}(\mathfrak{p}^m). \tag{43}$$

From the last two rows and last three columns we get

$$g_1(I_n - D) + g_2 B \in M_n(\mathfrak{p}^m), \qquad g_1 D - g_2 B \in I_n + M_n(\mathfrak{p}^m), \qquad g_2 \in M_n(\mathfrak{p}^m), \qquad (44)$$

$$g_3(I_n - D) + g_4 B \in M_n(\mathfrak{p}^m), \qquad g_3 D - g_4 B \in M_n(\mathfrak{p}^m), \qquad g_4 \in I_n + M_n(\mathfrak{p}^m), \quad (45)$$

where $M_n(\mathfrak{p}^m)$ is the set of $n \times n$ matrices with entries in $\mathfrak{p}^m$. Hence $g \in I_{2n} + M_{2n}(\mathfrak{p}^m) = \Gamma_{2n}(\mathfrak{p}^m)$. Multiplying (43) from the left by p^{-1} and looking at the lower left block, we see that $\left[\begin{smallmatrix} -C & I_n - D \\ A - I_n & B \end{smallmatrix} \right] \in M_{2n}(\mathfrak{p}^m)$, i.e., $h \in \Gamma_{2n}(\mathfrak{p}^m)$. Then it follows from (43) that $p \in \Gamma_{4n}(\mathfrak{p}^m)$. ∎

Let m be a *positive* integer such that $\chi|_{(1+\mathfrak{p}^m)\cap\mathfrak{o}^\times} = 1$. Let $f(g, s)$ be the unique function on $G_{4n}(F) \times \mathbb{C}$ such that

i) $f(pk, s) = \chi(p)\delta_{P_{4n}}(p)^{s+\frac{1}{2}}$ for all $p \in P_{4n}(F)$ and $k \in \Gamma_{4n}(\mathfrak{p}^m)$.

ii) $f(g, s) = 0$ if $g \notin P_{4n}(F)\Gamma_{4n}(\mathfrak{p}^m)$.

It is easy to see that f is well-defined. Evidently, $f \in I(\chi, s)$. Furthermore, for each $h \in G_{4n}(\mathfrak{o})$, define $f^{(h)} \in I(\chi, s)$ by the equation

$$f^{(h)}(g, s) = f(gh^{-1}, s).$$

4.3 Proposition. *Let π be any irreducible admissible representation of $G_{2n}(F)$. Let m be a positive integer such that $\chi|_{(1+\mathfrak{p}^m)\cap\mathfrak{o}^\times} = 1$ and such that there exists a vector ϕ in π fixed by $\Gamma_{2n}(\mathfrak{p}^m)$. Let $\tau \in \mathfrak{o}^\times$ and $Q_\tau := \left[\begin{smallmatrix} \tau I_{2n} & \\ & I_{2n} \end{smallmatrix} \right] Q_n \left[\begin{smallmatrix} \tau^{-1} I_{2n} & \\ & I_{2n} \end{smallmatrix} \right]$. Then $Z(s; f^{(Q_\tau)}, \phi) = \chi(\tau)^{-n} r_m \phi$, where $r_m = \mathrm{vol}(\Gamma_{2n}(\mathfrak{p}^m))$ is a non-zero rational number depending only on m.*

Proof. By definition,

$$Z(s; f^{(Q_\tau)}, \phi) = \int_{\mathrm{Sp}_{2n}(F)} f(Q_n \cdot (h, 1)Q_\tau^{-1}, s)\pi(h)\phi \, dh.$$

The integrand vanishes unless

$$Q_n \cdot (h, 1)Q_\tau^{-1} = p\gamma \qquad (46)$$

for some $p \in P_{4n}(F)$ and $\gamma \in \Gamma_{4n}(\mathfrak{p}^m)$. Put $t = \left[\begin{smallmatrix} \tau I_n & \\ & I_n \end{smallmatrix} \right]$. Then (46) is equivalent to

$$(Q_n(t, t)Q_n^{-1})(t^{-1}, t^{-1})p^{-1}Q_n(h, 1)Q_n^{-1} = \gamma'$$

for some $\gamma' \in \Gamma_{4n}(\mathfrak{p}^m)$. Put $p' = (Q_n(t, t)Q_n^{-1})(t^{-1}, t^{-1})p^{-1}$. By the last part of Lemma 2.2, $p' \in P_{4n}(F)$ with $\chi(p') = \chi(p)^{-1}\chi(\tau)^{-n}$. Now, using Lemma 4.2, we conclude that $\chi(p) = \chi(\tau)^{-n}$ and $h \in \Gamma_{2n}(\mathfrak{p}^m)$. So we obtain

$$\int_{\mathrm{Sp}_{2n}(F)} f(Q_n(h, 1)Q^{-1}, s)\pi(h)\phi \, dh = \chi(\tau)^{-n} \int_{\Gamma_{2n}(\mathfrak{p}^m)} \pi(h)\phi \, dh. \qquad (47)$$

This last integral equals $\mathrm{vol}(\Gamma_{2n}(\mathfrak{p}^m))\phi$, completing the proof. ∎

We end this subsection with an alternate definition of the function $f(g, s)$ defined above. Define the Siegel-type congruence subgroup $\Gamma_{0,4n}(\mathfrak{p}^m) = \{g \in \mathrm{Sp}_{4n}(\mathfrak{o}) : g \equiv \left[\begin{smallmatrix} * & * \\ 0_{2n} & * \end{smallmatrix}\right] \bmod \mathfrak{p}^m\}$, and for any $k \in \Gamma_{0,4n}(\mathfrak{p}^m)$, we denote $\chi(k) = \chi(\det(A))$ if $k = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. It is easy to check now that $f(g, s)$ is the unique function such that

i) $f(pk, s) = \chi(p)\chi(k)\delta_{P_{4n}}(p)^{s+\frac{1}{2}}$ for all $p \in P_{4n}(F)$ and $k \in \Gamma_{0,4n}(\mathfrak{p}^m)$.

ii) $f(g, s) = 0$ if $g \notin P_{4n}(F)\Gamma_{0,4n}(\mathfrak{p}^m)$.

The main difference between the above description and our original definition is that it uses the Siegel type congruence subgroup rather than the principal congruence subgroup. The fact that makes this alternate description possible is that $P_{4n}(F)\Gamma_{0,4n}(\mathfrak{p}^m) = P_{4n}(F)\Gamma_{4n}(\mathfrak{p}^m)$. In particular, this shows that our local section $f(g, s)$ is essentially identical to that used by Shimura [46, 49] in his work on Eisenstein series, which will be a key point for us later on.

5 The local integral at a real place

5.1 Holomorphic discrete series representations

We provide some preliminaries on holomorphic discrete series representations of $\mathrm{Sp}_{2n}(\mathbb{R})$. We fix the standard maximal compact subgroup

$$K = K^{(n)} = \{\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \in \mathrm{GL}_{2n}(\mathbb{R}) : A\,{}^tA + B\,{}^tB = 1,\ A\,{}^tB = B\,{}^tA\}.$$

We have $K \cong U(n)$ via $\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \mapsto A + iB$. Let

$$j(g, Z) = \det(CZ + D) \qquad \text{for } g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}_{2n}(\mathbb{R})$$

and Z in the Siegel upper half space $\mathbb{H}_n$. Then, for any integer k, the map

$$K \ni g \longmapsto j(g, I)^{-k}, \qquad \text{where } I = iI_n, \tag{48}$$

is a character of K.

Let $\mathfrak{h}$ be the compact Cartan subalgebra and $e_1, \ldots, e_n$ the linear forms on the complexification $\mathfrak{h}_{\mathbb{C}}$ defined in [3]. A system of positive roots is given by $e_i \pm e_j$ for $1 \leq i < j \leq n$ and $2e_j$ for $1 \leq j \leq n$. The positive compact roots are the $e_i - e_j$ for $1 \leq i < j \leq n$. The K-types are parametrized by the analytically integral elements $k_1e_1 + \ldots + k_ne_n$, where the k_i are integers with $k_1 \geq \ldots \geq k_n$. We write $\rho_{\mathbf{k}}$, $\mathbf{k} = (k_1, \ldots, k_n)$, for the K-type with highest weight $k_1e_1 + \ldots + k_ne_n$. If $k_1 = \ldots = k_n = k$, then $\rho_{\mathbf{k}}$ is the K_∞-type given in (48); we simply write ρ_k in this case.

The holomorphic discrete series representations of $\mathrm{Sp}_{2n}(\mathbb{R})$ are parametrized by elements $\lambda = \ell_1e_1 + \ldots \ell_ne_n$ with integers $\ell_1 > \ldots > \ell_n > 0$. The representation corresponding to the *Harish-Chandra parameter* λ contains the K-type $\rho_{\mathbf{k}}$, where $\mathbf{k} = \lambda + \sum_{j=1}^n je_j$, with multiplicity one; see Theorem 9.20 of [27]. We denote this representation by $\dot{\pi}_\lambda$ or by $\pi_{\mathbf{k}}$; sometimes one or the other notation is more convenient. If $\mathbf{k} = (k, \ldots, k)$ with a positive integer $k > n$, then we also write π_k for $\pi_{\mathbf{k}}$.

Let $G_{2n}(\mathbb{R})^+$ be the index two subgroup of $G_{2n}(\mathbb{R})$ consisting of elements with positive multiplier. We may extend a holomorphic discrete series representation $\dot{\pi}_\lambda$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ in a

trivial way to $G_{2n}(\mathbb{R})^+ \cong \mathrm{Sp}_{2n}(\mathbb{R}) \times \mathbb{R}_{>0}$. This extension induces irreducibly to $G_{2n}(\mathbb{R})$. We call the result a holomorphic discrete series representation of $G_{2n}(\mathbb{R})$ and denote it by the same symbol $\dot{\pi}_\lambda$ (or $\pi_{\mathbf{k}}$). These are the archimedean components of the automorphic representations corresponding to vector-valued holomorphic Siegel modular forms of degree n.

5.1 Lemma. *Let $\lambda = \ell_1 e_1 + \ldots + \ell_n e_n$ with integers $\ell_1 > \ldots > \ell_n > 0$.*

i) The holomorphic discrete series representation $\dot{\pi}_\lambda$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ embeds into

$$\mathrm{sgn}^{\ell_n+n}|\cdot|^{\ell_n} \times \ldots \times \mathrm{sgn}^{\ell_1+1}|\cdot|^{\ell_1} \rtimes 1, \tag{49}$$

and in no other principal series representation of $\mathrm{Sp}_{2n}(\mathbb{R})$.

ii) The holomorphic discrete series representation $\dot{\pi}_\lambda$ of $G_{2n}(\mathbb{R})$ embeds into

$$\mathrm{sgn}^{\ell_n+n}|\cdot|^{\ell_n} \times \ldots \times \mathrm{sgn}^{\ell_1+1}|\cdot|^{\ell_1} \rtimes \mathrm{sgn}^\varepsilon|\cdot|^{-\frac{1}{2}(\ell_1+\ldots+\ell_n)}, \tag{50}$$

where ε can be either 0 or 1, and in no other principal series representation of $G_{2n}(\mathbb{R})$.

Proof. i) follows from the main result of [55]. Part ii) can be deduced from i), observing that the holomorphic discrete series representations of $G_{2n}(\mathbb{R})$ are invariant under twisting by the sign character. ∎

5.2 Lemma. *Let k be a positive integer. Consider the degenerate principal series representation of $G_{2n}(\mathbb{R})$ given by*

$$J(s) := \mathrm{sgn}^k|\cdot|^{s-\frac{n+1}{2}} \rtimes \mathrm{sgn}^\varepsilon|\cdot|^{\frac{n(n+1)}{4}-\frac{ns}{2}}, \tag{51}$$

where $\varepsilon \in \{0,1\}$. Then $J(s)$ contains the holomorphic discrete series representation π_k of $G_{2n}(\mathbb{R})$ as a subrepresentation if and only if $s = k$.

Proof. By infinitesimal character considerations, we only need to prove the "if" part. Since π_k is invariant under twisting by sgn, we may assume that $\varepsilon = 0$. Consider the Borel-induced representation

$$J'(s) := \mathrm{sgn}^k|\cdot|^{s-n} \times \mathrm{sgn}^k|\cdot|^{s-n+1} \times \ldots \times \mathrm{sgn}^k|\cdot|^{s-1} \rtimes |\cdot|^{\frac{n(n+1)}{4}-\frac{ns}{2}}. \tag{52}$$

By Lemma 5.1 ii), π_k is a subrepresentation of $J'(k)$. Since $|\cdot|^{s-n} \times |\cdot|^{s-n+1} \times \ldots \times |\cdot|^{s-1}$ contains the trivial representation of $\mathrm{GL}_n(\mathbb{R})$ twisted by $|\cdot|^{s-\frac{n+1}{2}}$, it follows that $J(s) \subset J'(s)$. Let f_s be the function on $G_{2n}(\mathbb{R})$ given by

$$f_s\left(\begin{bmatrix} A & * \\ & u\,{}^t A^{-1} \end{bmatrix} g, s\right) = \mathrm{sgn}^k(\det(A))|u|^{-\frac{ns}{2}}|\det(A)|^s j(g,I)^{-k} \tag{53}$$

for $A \in \mathrm{GL}_n(\mathbb{R})$, $u \in \mathbb{R}^\times$ and $g \in K$. Then f_s is a well-defined element of $J(s)$. Since f_s is the unique up to multiples vector of weight k in $J'(s)$, π_k lies in the subspace $J(k)$ of $J'(k)$. ∎

Our method to calculate the local archimedean integrals (35) will work for holomorphic discrete series representations $\dot{\pi}_\lambda$, where $\lambda = \ell_1 e_1 + \ldots + \ell_n e_n$ with $\ell_1 > \ldots > \ell_n > 0$ satisfies

$$\ell_j - \ell_{j-1} \text{ is odd for } 2 \leq j \leq n. \tag{54}$$

Equivalently, we work with the holomorphic discrete series representations $\pi_{\mathbf{k}}$, where $\mathbf{k} = k_1 e_1 + \ldots + k_n e_n$ with $k_1 \geq \ldots \geq k_n > n$ and all k_i of the same parity; this last condition can be seen to be equivalent to (54). An example for λ satisfying (54) is $(k-1)e_1 + \ldots + (k-n)e_n$, the Harish-Chandra parameter of π_k. The next lemma implies that whenever (54) is satisfied, then $\dot{\pi}_\lambda$ contains a convenient scalar K-type to work with.

5.3 Lemma. *Assume that $\lambda = \ell_1 e_1 + \ldots + \ell_n e_n$ with $\ell_1 > \ldots > \ell_n > 0$ satisfying (54). Let m be a non-negative, even integer. Then the holomorphic discrete series representation $\dot{\pi}_\lambda$ contains the K-type*

$$\rho_{\mathbf{m}}, \qquad \mathbf{m} = (\ell_1 + 1 + m)(e_1 + \ldots + e_n), \tag{55}$$

with multiplicity one.

Proof. By Theorem 8.1 of [27] we need only show that $\rho_{\mathbf{m}}$ occurs in $\dot{\pi}_\lambda$. We will use induction on n. The result is obvious for $n = 1$. Assume that $n > 1$, and that the assertion has already been proven for $n - 1$.

Using standard notations as in [3], we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^+$. The universal enveloping algebra of $\mathfrak{p}_{\mathbb{C}}^+$ is isomorphic to the symmetric algebra $S(\mathfrak{p}_{\mathbb{C}}^+)$. We have $\dot{\pi}_\lambda \cong S(\mathfrak{p}_{\mathbb{C}}^+) \otimes \rho_\lambda$ as K-modules. Let I be the subalgebra of $S(\mathfrak{p}_{\mathbb{C}}^+)$ spanned by the highest weight vectors of its K-types. By Theorem A of [26], there exists in I an element D_+ of weight $2(e_1 + \ldots + e_n)$. By the main result of [24], the space of K-highest weight vectors of $\dot{\pi}_\lambda$ is acted upon freely by I. It follows that we need to prove our result only for $m = 0$.

We will use the Blattner formula proven in [21]. It says that the multiplicity with which $\rho_{\mathbf{m}}$ occurs in $\dot{\pi}_\lambda$ is given by

$$\mathrm{mult}_\lambda(\mathbf{m}) = \sum_{w \in W_K} \varepsilon(w) Q(w(\mathbf{m} + \rho_c) - \lambda - \rho_n). \tag{56}$$

Here, W_K is the compact Weyl group, which in our case is isomorphic to the symmetric group S_n, and ε is the sign character on W_K; the symbols ρ_c and ρ_n denote the half sums of the positive compact and non-compact roots, respectively; and $Q(\mu)$ is the number of ways to write μ as a sum of positive non-compact roots. In our case

$$\rho_c = \frac{1}{2} \sum_{j=1}^n (n+1-2j) e_j, \qquad \rho_n = \frac{n+1}{2} \sum_{j=1}^n e_j. \tag{57}$$

Hence

$$\mathrm{mult}_\lambda(\mathbf{m}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) Q\Big(\sigma\Big((\ell_1 + 1 + m)\sum_{j=1}^n e_j + \frac{1}{2}\sum_{j=1}^n (n+1-2j)e_j\Big) - \lambda - \frac{n+1}{2}\sum_{j=1}^n e_j\Big)$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) Q\Big(\sum_{j=1}^n (\ell_1 + 1 + m - \ell_j)e_j - \sigma\Big(\sum_{j=1}^n j e_j\Big)\Big). \tag{58}$$

Now assume that $m = 0$. Then

$$\mathrm{mult}_\lambda(\mathbf{m}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) Q\Big(\sum_{j=1}^n (\ell_1 + 1 - \ell_j)e_j - \sigma\Big(\sum_{j=1}^n j e_j\Big)\Big). \tag{59}$$

If $\sigma(1) \neq 1$, then the coefficient of e_1 is negative, implying that $Q(\ldots) = 0$. Hence

$$\mathrm{mult}_\lambda(\mathbf{m}) = \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \varepsilon(\sigma) Q\Big(\sum_{j=2}^n (\ell_1 + 1 - \ell_j) e_j - \sigma\Big(\sum_{j=2}^n j e_j \Big) \Big)$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \varepsilon(\sigma) Q\Big(\sum_{j=1}^{n-1} (\ell_1 - \ell_{j+1}) e_{j+1} - \sigma\Big(\sum_{j=1}^{n-1} j e_{j+1} \Big) \Big). \qquad (60)$$

If we set $e'_j = e_{j+1}$ and $m' = \ell_1 - \ell_2 - 1$, then this can be written as

$$\mathrm{mult}_\lambda(\mathbf{m}) = \sum_{\sigma \in S_{n-1}} \varepsilon(\sigma) Q\Big(\sum_{j=1}^{n-1} (\ell_2 + 1 + m' - \ell_{j+1}) e'_j - \sigma\Big(\sum_{j=1}^{n-1} j e'_j \Big) \Big). \qquad (61)$$

We see that this is the formula (58), with $n-1$ instead of n and m' instead of m; note that m' is even and non-negative by our hypotheses. There are two different Q-functions involved, for n and for $n-1$, but since the argument of Q in (60) has no e_1, we may think of it as the Q-function for $n-1$. Therefore (61) represents the multiplicity of the $K^{(n-1)}$-type $(\ell_2 + 1 + m')(e'_1 + \ldots + e'_{n-1})$ in the holomorphic discrete series representation $\dot\pi_{\lambda'}$ of $\mathrm{Sp}_{2(n-1)}(\mathbb{R})$, where $\lambda' = \ell_2 e'_1 + \ldots + \ell_n e'_{n-1}$. By induction hypothesis, this multiplicity is 1, completing our proof.   ∎

5.2   Calculating the integral

In the remainder of this section we fix a real place v and calculate the local archimedean integral (35) for a certain choice of vectors f and w. To ease notation, we omit the subscript v. We assume that the underlying representation π of $G_{2n}(\mathbb{R})$ is a holomorphic discrete series representation $\dot\pi_\lambda$, where $\lambda = \ell_1 e_1 + \ldots + \ell_n e_n$ with $\ell_1 > \ldots > \ell_n > 0$ satisfies (54). Set $k = \ell_1 + 1$. By Lemma 5.3, the K-type ρ_k appears in π with multiplicity 1. Let w_λ be a vector spanning this one-dimensional K-type. We choose $w = w_\lambda$ as our vector in the zeta integral $Z(s, f, w)$.

To explain our choice of f, let $J(s)$ be the degenerate principal series representation of $G_{4n}(\mathbb{R})$ defined in Lemma 5.2 (hence, we replace n by $2n$ in (51)). We see from (17) that $I(\mathrm{sgn}^k, s)$ equals $J((2n+1)(s + \frac{1}{2}))$ for appropriate $\varepsilon \in \{0, 1\}$. Let $f_k(\cdot, s)$ be the vector spanning the K-type $K^{(2n)} \ni g \longmapsto j(g, I)^{-k}$ and normalized by $f_k(1, s) = 1$. Explicitly,

$$f_k\big(\big[\begin{smallmatrix} A & * \\ & u\, ^tA^{-1} \end{smallmatrix} \big] g, s \big) = \mathrm{sgn}^k(\det(A))\, \mathrm{sgn}^{nk}(u)\, |u^{-n} \det(A)|^{(2n+1)(s+\frac{1}{2})}\, j(g, I)^{-k} \qquad (62)$$

for $A \in \mathrm{GL}_{2n}(\mathbb{R})$, $u \in \mathbb{R}^\times$ and $g \in K^{(2n)}$. Then $f = f_k$ is the section which we will put in our local archimedean integral $Z(s, f, w)$.

Thus consider $Z(s, f_k, w_\lambda)$. By Proposition 3.2 i), this integral is a vector in π. The observation (19), together with the transformation properties of f_k, imply that for $g \in K$

$$\pi(g) Z(s, f_k, w_\lambda) = j(g, I)^{-k} Z(s, f_k, w_\lambda). \qquad (63)$$

Since the K-type ρ_k occurs only once in π, it follows that

$$Z(s, f_k, w_\lambda) = B_\lambda(s) w_\lambda \qquad (64)$$

for a constant $B_\lambda(s)$ depending on s and the Harish-Chandra parameter λ. The rest of this section is devoted to calculating $B_\lambda(s)$ explicitly.

The scalar minimal K-type case

We first consider the case $\lambda = (k-1)e_1 + \ldots + (k-n)e_n$ with $k > n$. Then $\dot{\pi}_\lambda = \pi_k$, the holomorphic discrete series representation of $G_{2n}(\mathbb{R})$ with minimal K-type ρ_k. Let w_k be a vector spanning this minimal K-type. Let $\langle \, , \, \rangle$ be an appropriately normalized invariant hermitian inner product on the space of π_k such that $\langle w_k, w_k \rangle = 1$. As proved in the appendix of [28], we have the following simple formula for the corresponding matrix coefficient,

$$\langle \pi_k(h)w_k, w_k \rangle = \begin{cases} \dfrac{\mu_n(h)^{nk/2}\, 2^{nk}}{\det(A + D + i(C - B))^k} & \text{for } \mu(h) > 0, \\ 0 & \text{for } \mu(h) < 0. \end{cases} \tag{65}$$

Here, $h = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G_{2n}(\mathbb{R})$. We will need the following result.

5.4 Lemma. *For a complex number z, let*

$$\gamma_n(z) = \int_T \left(\prod_{1 \le i < j \le n} (t_i^2 - t_j^2) \right) \left(\prod_{j=1}^n t_j \right)^{-z-n} d\mathbf{t}, \tag{66}$$

where $d\mathbf{t}$ is the Lebesgue measure and

$$T = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \ : \ t_1 > \ldots > t_n > 1\}. \tag{67}$$

Then, for real part of z large enough,

$$\gamma_n(z) = \prod_{m=1}^n (m-1)! \prod_{j=1}^m \frac{1}{z - m - 1 + 2j}. \tag{68}$$

Proof. After some straightforward variable transformations, our integral reduces to the Selberg integral; see [41] or [14]. ∎

5.5 Proposition. *Assume that $k > n$ and $\lambda = (k-1)e_1 + \ldots + (k-n)e_n$, so that $\dot{\pi}_\lambda = \pi_k$. Then*

$$B_\lambda(s) = \frac{\pi^{n(n+1)/2}}{\prod_{m=1}^n (m-1)!}\, i^{nk}\, 2^{-n(2n+1)s + 3n/2}\, \gamma_n\left((2n+1)s - \frac{1}{2} + k\right), \tag{69}$$

where γ_n is the rational function (68).

Proof. Taking the inner product with w_k on both sides of (64), we obtain

$$B_\lambda(s) = \int_{\mathrm{Sp}_{2n}(\mathbb{R})} f_k(Q_n \cdot (h, 1), s) \langle \pi_k(h)w_k, w_k \rangle \, dh. \tag{70}$$

Since the integrand is left and right K-invariant, we may apply the integration formula (27). Thus

$$B_\lambda(s) = \alpha_n \int_{\mathfrak{a}^+} \left(\prod_{\lambda \in \Sigma^+} \sinh(\lambda(H)) \right) f_k(Q_n \cdot (\exp(H), 1), s) \langle \pi_k(\exp(H))w_k, w_k \rangle \, dH. \tag{71}$$

This time we work with the positive system $\Sigma^+ = \{e_i - e_j : 1 \le i < j \le n\} \cup \{e_i + e_j : 1 \le i \le j \le n\}$, so that

$$\mathfrak{a}^+ = \{\text{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n) : a_1 > \ldots > a_n > 0\}. \tag{72}$$

The function f_k in (71) can be evaluated as follows,

$$f_k(Q_n \cdot (\exp(H), 1), s) = f_k(\begin{bmatrix} I_n & & & \\ & I_n & & \\ & I_n & I_n & \\ I_n & & & I_n \end{bmatrix} \begin{bmatrix} I_n & & & \\ & & & I_n \\ & & I_n & \\ & -I_n & & \end{bmatrix} \cdot (\exp(H), 1), s)$$

$$= i^{nk} f_k(\begin{bmatrix} I_n & & & \\ & I_n & & \\ & I_n & I_n & \\ I_n & & & I_n \end{bmatrix} \begin{bmatrix} A & & & \\ & I_n & & \\ & & A^{-1} & \\ & & & I_n \end{bmatrix}, s) \qquad (A = \begin{bmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n} \end{bmatrix})$$

$$= i^{nk} e^{(a_1 + \ldots + a_n)(2n+1)(s+\frac{1}{2})} f_k(\begin{bmatrix} I_n & & & \\ & I_n & & \\ & A & I_n & \\ A & & & I_n \end{bmatrix}, s).$$

Now use

$$\begin{bmatrix} 1 & \\ e^{a_j} & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix} r(\theta), \qquad r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{73}$$

with

$$x = \frac{e^{a_j}}{1 + e^{2a_j}}, \qquad y = \frac{1}{1 + e^{2a_j}}, \qquad e^{i\theta} = \frac{1 - ie^{a_j}}{(1 + e^{2a_j})^{1/2}}. \tag{74}$$

It follows that

$$f_k(Q_n \cdot (\exp(H), 1), s) = i^{nk} \prod_{j=1}^{n} (e^{a_j} + e^{-a_j})^{-(2n+1)(s+\frac{1}{2})}. \tag{75}$$

Substituting this and (65) into (71), we get after some simplification

$$\alpha_n^{-1} B_\lambda(s) = i^{nk} 2^{-n(2n+1)(s+\frac{1}{2})+n}$$

$$\times \int_{\mathfrak{a}^+} \prod_{1 \le i < j \le n} (\cosh(a_i)^2 - \cosh(a_j)^2) \prod_{j=1}^{n} \cosh(a_j)^{-(2n+1)s+\frac{1}{2}-n-k} \prod_{j=1}^{n} \sinh(a_j) \, dH.$$

Now introduce the new variables $t_j = \cosh(a_j)$. The domain $\mathfrak{a}^+$ turns into the domain T defined in (67). We get

$$B_\lambda(s) = \alpha_n i^{nk} 2^{-n(2n+1)(s+\frac{1}{2})+n} \int_T \left(\prod_{1 \le i < j \le n} (t_i^2 - t_j^2) \right) \left(\prod_{j=1}^{n} t_j \right)^{-(2n+1)s+\frac{1}{2}-n-k} d\mathbf{t}. \tag{76}$$

Thus our assertion follows from Lemma 5.4 and the value of α_n given in (130). ∎

Calculation of $B_\lambda(s)$ for general Harish-Chandra parameter

Now consider a general $\lambda = \ell_1 e_1 + \ldots + \ell_n e_n$ for which the condition (54) is satisfied. By (50), we may embed $\dot{\pi}_\lambda$ into

$$\text{sgn}^k |\cdot|^{\ell_n} \times \ldots \times \text{sgn}^k |\cdot|^{\ell_1} \rtimes |\cdot|^{-\frac{1}{2}(\ell_1 + \ldots + \ell_n)}. \tag{77}$$

In this induced model, the weight k vector w_λ in $\dot{\pi}_\lambda$ has the formula

$$w_\lambda\left(\begin{bmatrix} a_1 & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & & \vdots \\ & & a_n & * & \cdots & * \\ & & & a_0 a_1^{-1} & & \\ & & & \vdots & \ddots & \\ & & & * & \cdots & a_0 a_n^{-1} \end{bmatrix} g\right) = \mathrm{sgn}(a_1 \cdot \ldots \cdot a_n)^k \, |a_0|^{-\frac{1}{2}(\ell_1 + \ldots + \ell_n) - \frac{n(n+1)}{4}}$$

$$\left(\prod_{j=1}^{n} |a_j|^{\ell_{n+1-j} + n + 1 - j}\right) j(g, I)^{-k} \tag{78}$$

for $a_1, \ldots, a_n \in \mathbb{R}^\times$ and $g \in K^{(n)}$. Evaluating (64) at 1, we get

$$B_\lambda(s) = \int\limits_{\mathrm{Sp}_{2n}(\mathbb{R})} f_k(Q_n \cdot (h, 1), s) w_\lambda(h) \, dh. \tag{79}$$

Recall the beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{80}$$

One possible integral representation for the beta function is

$$B(x, y) = \int\limits_0^\infty \frac{a^{x-1}}{(a+1)^{x+y}} \, da = 2 \int\limits_0^\infty \frac{a^{x-y}}{(a+a^{-1})^{x+y}} \, d^\times a \qquad \text{for } \mathrm{Re}(x), \mathrm{Re}(y) > 0. \tag{81}$$

For $s \in \mathbb{C}$ and $m \in \mathbb{Z}$, let

$$\beta(m, s) = B\left(\left(n + \frac{1}{2}\right)s + \frac{1}{4} + \frac{m}{2}, \left(n + \frac{1}{2}\right)s + \frac{1}{4} - \frac{m}{2}\right). \tag{82}$$

5.6 Lemma. *We have*

$$B_\lambda(s) = i^{nk}\left(\prod_{j=1}^{n} \beta(\ell_j, s)\right) C_k(s), \tag{83}$$

where

$$C_k(s) = \int\limits_{\tilde{N}_1} \int\limits_{\tilde{N}_2} f_k\left(\begin{bmatrix} I_n & & & \\ & I_n & & \\ & {}^t V & I_n & \\ V & X & & I_n \end{bmatrix}, s\right) dX \, dV \tag{84}$$

depends only on $k = \ell_1 + 1$. Here, $\tilde{N}_1$ is the space of upper triangular nilpotent matrices of size $n \times n$, and $\tilde{N}_2$ is the space of symmetric $n \times n$ matrices.

Proof. In this proof we write $f_k(h)$ instead of $f_k(h, s)$ for simplicity. We will use the Iwasawa decomposition to calculate the integral (79). By Proposition A.3, the relevant integration formula for right $K^{(n)}$-invariant functions φ is

$$\int\limits_{\mathrm{Sp}_{2n}(\mathbb{R})} \varphi(h) \, dh = 2^n \int\limits_A \int\limits_N \varphi(an) \, dn \, da, \tag{85}$$

where N is the unipotent radical of the Borel subgroup, dn is the Lebesgue measure, $A = \{\operatorname{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) : a_1, \ldots, a_n > 0\}$, and $da = \frac{da_1}{a_1} \ldots \frac{da_n}{a_n}$. Thus, using (78),

$$B_\lambda(s) = 2^n \int\limits_A \int\limits_N f_k(Q_n \cdot (an, 1)) w_\lambda(an) \, dn \, da$$

$$= 2^n \int\limits_A \int\limits_N \left(\prod_{j=1}^n a_j^{\ell_{n+1-j}+n+1-j} \right) f_k(Q_n \cdot (an, 1)) \, dn \, da. \tag{86}$$

We have $N = N_1 N_2$, where

$$N_1 = \left\{ \begin{bmatrix} U \\ & {}^t U^{-1} \end{bmatrix} : U \text{ upper triangular unipotent} \right\}, \qquad N_2 = \left\{ \begin{bmatrix} I_n & X \\ & I_n \end{bmatrix} : X \text{ symmetric} \right\}. \tag{87}$$

Write further $a = \begin{bmatrix} D \\ & D^{-1} \end{bmatrix}$ with $D = \operatorname{diag}(a_1, \ldots, a_n)$. Then

$$f_k(Q_n \cdot (an, 1)) = f_k\left(\begin{bmatrix} I_n \\ I_n & I_n \\ I_n & & I_n \end{bmatrix} \begin{bmatrix} I_n \\ & I_n \\ -I_n & & I_n \end{bmatrix} \begin{bmatrix} D \\ & I_n \\ & & D^{-1} & I_n \end{bmatrix} \begin{bmatrix} U \\ & I_n \\ & {}^t U^{-1} & I_n \end{bmatrix} \begin{bmatrix} I_n & -X \\ & I_n \\ & & I_n & I_n \end{bmatrix} \right)$$

$$= i^{nk} |a_1 \cdot \ldots \cdot a_n|^{(2n+1)(s+\frac{1}{2})} f_k\left(\begin{bmatrix} I_n \\ I_n \\ D & I_n \\ D & & I_n \end{bmatrix} \begin{bmatrix} U \\ & I_n \\ & {}^t U^{-1} & I_n \end{bmatrix} \begin{bmatrix} I_n & -X \\ & I_n \\ & & I_n & I_n \end{bmatrix} \right). \tag{88}$$

We will use the identity

$$\begin{bmatrix} 1 \\ a_j & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_j \\ & 1 \end{bmatrix} \begin{bmatrix} y_j^{1/2} \\ & y_j^{-1/2} \end{bmatrix} r(\theta_i) \tag{89}$$

with

$$x_j = \frac{a_j}{1 + a_j^2}, \qquad y_j = \frac{1}{1 + a_j^2}, \qquad e^{i\theta_j} = \frac{1 - ia_j}{(1 + a_j^2)^{1/2}} \tag{90}$$

Let

$$Y = \begin{bmatrix} y_1^{1/2} \\ & \ddots \\ & & y_n^{1/2} \end{bmatrix}, \qquad C = \begin{bmatrix} \cos(\theta_1) \\ & \ddots \\ & & \cos(\theta_n) \end{bmatrix}, \qquad S = \begin{bmatrix} \sin(\theta_1) \\ & \ddots \\ & & \sin(\theta_n) \end{bmatrix}. \tag{91}$$

Then

$$\begin{bmatrix} I_n \\ I_n \\ D & I_n \\ D & & I_n \end{bmatrix} = \begin{bmatrix} Y \\ & Y & * \\ & * & Y^{-1} \\ & & & Y^{-1} \end{bmatrix} \begin{bmatrix} C & S \\ & C & S \\ -S & & C \\ -S & & & C \end{bmatrix}, \tag{92}$$

and thus

$$f_k(Q_n \cdot (an, 1)) = i^{nk} \left(\prod_{j=1}^n \frac{1}{a_j + a_j^{-1}} \right)^{(2n+1)(s+\frac{1}{2})} f_k\left(\begin{bmatrix} C & S \\ & C & S \\ -S & & C \\ -S & & & C \end{bmatrix} \begin{bmatrix} U \\ & I_n \\ & {}^t U^{-1} & I_n \end{bmatrix} \begin{bmatrix} I_n & -X \\ & I_n \\ & & I_n & I_n \end{bmatrix} \right). \tag{93}$$

Write $U = I_n + V$, so that V is upper triangular nilpotent. A calculation confirms that

$$\begin{bmatrix} C & S \\ & C & S \\ -S & & C \\ -S & & & C \end{bmatrix} \begin{bmatrix} U \\ & I_n \\ & {}^t U^{-1} & I_n \end{bmatrix} = p \begin{bmatrix} I_n \\ & I_n \\ {}^t Z & I_n \\ Z & & I_n \end{bmatrix} \begin{bmatrix} C & S \\ & C & S \\ -S & & C \\ -S & & & C \end{bmatrix}, \tag{94}$$

where $Z = -(1 + SVS)^{-1}SVC$ and $p = \begin{bmatrix} B & \\ & {}^t B^{-1} \end{bmatrix}$ with $\det(B) = 1$. Hence

$$f_k(Q_n \cdot (an, 1)) = i^{nk} \left(\prod_{j=1}^{n} \frac{1}{a_j + a_j^{-1}} \right)^{(2n+1)(s+\frac{1}{2})} f_k\left(\begin{bmatrix} I_n & & \\ & I_n & \\ Z & {}^t Z \, I_n & I_n \end{bmatrix} \begin{bmatrix} C & & S \\ & C \, S & \\ -S & -S \, C & C \end{bmatrix} \begin{bmatrix} I_n & & -X \\ & I_n & \\ & & I_n \end{bmatrix} \right). \tag{95}$$

Let $\tilde{N}_1$ be the Euclidean space of upper triangular nilpotent real matrices of size $n \times n$. Then it is an exercise to verify that

$$\varphi \longmapsto \int_{\tilde{N}_1} \varphi(I_n + V) \, dV, \tag{96}$$

where dV is the Lebesgue measure, defines a Haar measure on the group of upper triangular unipotent real matrices. (Use the fact that $V \mapsto UV$ defines an automorphism of $\tilde{N}$ of determinant 1, for every upper triangular unipotent U.)

Therefore, as we integrate (95) over N_1, we may treat V as a Euclidean variable. We then have to consider the Jacobian of the change of variables $V \mapsto Z$. It is not difficult to show that this Jacobian is $\prod_{j=1}^{n} \sin(\theta_j)^{j-n} \cos(\theta_j)^{1-j}$. Substituting from (90), we find

$$\left| \prod_{j=1}^{n} \sin(\theta_j)^{j-n} \cos(\theta_j)^{1-j} \right| = \left(\prod_{j=1}^{n} a_j \right)^{-\frac{n+1}{2}} \left(\prod_{j=1}^{n} a_j^j \right) \left(\prod_{j=1}^{n} \frac{1}{a_j + a_j^{-1}} \right)^{\frac{1-n}{2}}. \tag{97}$$

Using the above and some more matrix identities, we get

$$\int_{N_1} \int_{N_2} f_k(Q_n \cdot (an_1 n_2, 1)) \, dn_1 \, dn_2 = i^{nk} \left(\prod_{j=1}^{n} \frac{1}{a_j + a_j^{-1}} \right)^{(2n+1)s + \frac{1}{2}} \left(\prod_{j=1}^{n} a_j \right)^{-n-1} \left(\prod_{j=1}^{n} a_j^j \right)$$

$$\int_{\tilde{N}_1} \int_{\tilde{N}_2} f_k\left(\begin{bmatrix} I_n & & \\ & I_n & \\ V & {}^t V \, I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & & \\ & I_n & \\ X & & I_n \, I_n \end{bmatrix} \right) dX \, dV. \tag{98}$$

This last integral is the $C_k(s)$ defined in (84). Going back to (86) and using (81), we have

$$B_\lambda(s) = 2^n \int_A \int_{N_1} \int_{N_2} \left(\prod_{j=1}^{n} a_j^{\ell_{n+1-j} + n + 1 - j} \right) f_k(Q_n \cdot (an_1 n_2, 1)) \, dn_2 \, dn_1 \, da$$

$$= 2^n i^{nk} \int_A \left(\prod_{j=1}^{n} a_j^{\ell_{n+1-j}} \right) \left(\prod_{j=1}^{n} \frac{1}{a_j + a_j^{-1}} \right)^{(2n+1)s + \frac{1}{2}} da \cdot C_k(s)$$

$$= i^{nk} \prod_{j=1}^{n} \left(B\left(\left(n + \frac{1}{2} \right)s + \frac{1}{4} + \frac{\ell_j}{2}, \left(n + \frac{1}{2} \right)s + \frac{1}{4} - \frac{\ell_j}{2} \right) \right) \cdot C_k(s). \tag{99}$$

This concludes the proof.   ∎

5.7 Lemma. *The function $C_k(s)$ defined in (84) is given by*

$$C_k(s) = \frac{\pi^{n(n+1)/2}}{\prod_{m=1}^{n}(m-1)!} 2^{-n(2n+1)s+3n/2} \frac{\gamma_n((2n+1)s - \frac{1}{2} + k)}{\prod_{j=1}^{n}\beta(k-j,s)}, \tag{100}$$

where γ_n is the rational function from Lemma 5.4, and $\beta(m,s)$ is defined in (82).

Proof. Consider $\lambda = (k-1)e_1 + \ldots + (k-n)e_n$. Then

$$B_\lambda(s) = \frac{\pi^{n(n+1)/2}}{\prod_{m=1}^{n}(m-1)!} i^{nk} 2^{-n(2n+1)s+3n/2} \gamma_n\left((2n+1)s - \frac{1}{2} + k\right), \tag{101}$$

by Proposition 5.5. On the other hand,

$$B_\lambda(s) = i^{nk}\left(\prod_{j=1}^{n}\beta(k-j,s)\right)C_k(s) \tag{102}$$

by Lemma 5.6. The assertion follows by comparing the two expressions. ∎

5.8 Proposition. *Let $\lambda = \ell_1 e_1 + \ldots + \ell_n e_n$, $\ell_1 > \ldots > \ell_n > 0$, be a Harish-Chandra parameter for which the condition (54) is satisfied. Denote $k_j = \ell_j + j$ and $k = k_1$. Let $\pi = \dot\pi_\lambda$ be the corresponding holomorphic discrete series representation of $G_{2n}(\mathbb{R})$ and let w_λ be a vector in π spanning the K-type ρ_k. Let f_k be the element of $I(\mathrm{sgn}^k, s)$ given by (62). Then*

$$Z(s, f_k, w_\lambda) = i^{nk}\,\pi^{n(n+1)/2}A_{\mathbf{k}}((2n+1)s - 1/2)\,w_\lambda$$

with the function $A_{\mathbf{k}}(z)$ is defined as

$$A_{\mathbf{k}}(z) = 2^{-n(z-1)}\left(\prod_{j=1}^{n}\prod_{i=1}^{j}\frac{1}{z+k-1-j+2i}\right)\left(\prod_{j=1}^{n}\prod_{i=0}^{\frac{k-k_j}{2}-1}\frac{z-(k-1-j-2i)}{z+(k-1-j-2i)}\right). \tag{103}$$

Proof. By Lemma 5.6 and Lemma 5.7, $Z(s, f_k, w_\lambda) = B_\lambda(s)w_\lambda$ with

$$B_\lambda(s) = i^{nk}\frac{\pi^{n(n+1)/2}}{\prod_{m=1}^{n}(m-1)!}2^{-n(2n+1)s+3n/2}\gamma_n\left((2n+1)s - \frac{1}{2} + k\right)\prod_{j=1}^{n}\frac{\beta(\ell_j,s)}{\beta(k-j,s)}. \tag{104}$$

Inductively one confirms the identity

$$B(x+m, y-m) = B(x,y)\prod_{i=0}^{m-1}\frac{x+i}{y-i-1} \tag{105}$$

for any integer $m \geq 0$. Use the abbreviation $t = (n+\frac{1}{2})s + \frac{1}{4}$. Applying the above formula with $m = \frac{k-k_j}{2}$, which by (54) is a non-negative integer, and replacing i by $m-1-i$, we get

$$\frac{\beta(\ell_j,s)}{\beta(k-j,s)} = \prod_{i=0}^{m-1}\frac{t - \frac{\ell_j}{2} - m + i}{t + \frac{\ell_j}{2} + m - 1 - i}. \tag{106}$$

The result follows by using formula (68) for γ_n. ∎

5.9 Remark. *Using (103), one can check that $A_{\mathbf{k}}(t)$ is a non-zero rational number for any integer t satisfying $0 \leq t \leq k_n - n$.*

6   The global integral representation

6.1   The main result

Consider the global field $F = \mathbb{Q}$ and its ring of adeles $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. All the results are easily generalizable to a totally real number field. Let $\pi \cong \otimes \pi_p$ be a cuspidal automorphic representation of $G_{2n}(\mathbb{A})$. We assume that π_{∞} is a holomorphic discrete series representation $\pi_{\mathbf{k}}$ with $\mathbf{k} = k_1 e_1 + \ldots + k_n e_n$, where $k_1 \geq \ldots \geq k_n > n$ and all k_i have the same parity. (From now on it is more convenient to work with the minimal K-type $\mathbf{k}$ rather than the Harish-Chandra parameter λ.) We set $k = k_1$. Let $\chi = \otimes \chi_p$ be a character of $\mathbb{Q}^{\times} \backslash \mathbb{A}^{\times}$ such that $\chi_{\infty} = \operatorname{sgn}^k$. Let $N = \prod_{p|N} p^{m_p}$ be an integer such that

- For each finite prime $p \nmid N$ both π_p and χ_p are unramified.

- For a prime $p|N$, we have $\chi_p|_{(1+p^{m_p}\mathbb{Z}_p) \cap \mathbb{Z}_p^{\times}} = 1$ and π_p has a vector ϕ_p that is right invariant under the principal congruence subgroup $\Gamma_{2n}(p^{m_p})$ of $\operatorname{Sp}_{2n}(\mathbb{Z}_p)$.

Let ϕ be a cusp form in the space of π corresponding to a pure tensor $\otimes \phi_p$, where the local vectors are chosen as follows. For $p \nmid N$ choose ϕ_p to be a spherical vector; for a $p|N$ choose ϕ_p to be a vector right invariant under $\Gamma_{2n}(p^{m_p})$; and for $p = \infty$ choose ϕ_{∞} to be a vector in π_{∞} spanning the K_{∞}-type ρ_k; see Lemma 5.3. Let $f = \otimes f_p \in I(\chi, s)$ be composed of the following local sections. For a finite prime $p \nmid N$ let f_p be the spherical vector normalized by $f_p(1) = 1$; for $p|N$ choose f_p as in Sect. 4.2 (with the positive integer m of that section equal to the m_p above); and for $p = \infty$, choose f_{∞} by (62). Define $L^N(s, \pi \boxtimes \chi, \varrho_{2n+1}) = \prod_{\substack{p \nmid N \\ p \neq \infty}} L(s, \pi_p \boxtimes \chi_p, \varrho_{2n+1})$, where the local factors on the right are given by (42).

Next, for any $h \in \prod_{p < \infty} \operatorname{Sp}_{4n}(\mathbb{Z}_p)$, define $f^{(h)}(g, s) = f(gh^{-1}, s)$. Let Q denote the element Q_n embedded diagonally in $\prod_{p < \infty} \operatorname{Sp}_{4n}(\mathbb{Z}_p)$, and for any $\tau \in \hat{\mathbb{Z}}^{\times} = \prod_{p < \infty} \mathbb{Z}_p$, let $Q_{\tau} = \begin{bmatrix} \tau I_{2n} & \\ & I_{2n} \end{bmatrix} Q \begin{bmatrix} \tau^{-1} I_{2n} & \\ & I_{2n} \end{bmatrix}$. We can now state our global integral representation.

6.1 Theorem. *Let the notation be as above. Then the function $L^N(s, \pi \boxtimes \chi, \varrho_{2n+1})$ can be analytically continued to a meromorphic function of s with only finitely many poles. Furthermore, for all $s \in \mathbb{C}$ and $g \in G_{2n}(\mathbb{A})$, and any $\tau \in \hat{\mathbb{Z}}^{\times}$,*

$$Z(s, f^{(Q_{\tau})}, \phi)(g) = \frac{L^N((2n+1)s + 1/2, \pi \boxtimes \chi, \varrho_{2n+1})}{L^N((2n+1)(s+1/2), \chi) \prod_{j=1}^{n} L^N((2n+1)(2s+1) - 2j, \chi^2)}$$
$$\times i^{nk} \chi(\tau)^{-n} \pi^{n(n+1)/2} \left(\prod_{p|N} \operatorname{vol}(\Gamma_{2n}(p^{m_p})) \right) A_{\mathbf{k}}((2n+1)s - 1/2)\phi(g), \quad (107)$$

with the rational function $A_{\mathbf{k}}(z)$ as defined as in Proposition 5.8.

Proof. By Theorem 3.6, Proposition 4.1, Proposition 4.3 and Proposition 5.8, the equation (107) is true for all $\operatorname{Re}(s)$ sufficiently large. Since the left side defines a meromorphic function of s for each g, it follows that the right side can be analytically continued to a meromorphic function of s such that (107) always holds. ∎

6.2   A classical reformulation

We now rewrite the above theorem in classical language. For any congruence subgroup Γ of $\mathrm{Sp}_{2n}(\mathbb{R})$ with the symmetry property $\begin{bmatrix} & -I_n \\ I_n & \end{bmatrix} \Gamma \begin{bmatrix} & -I_n \\ I_n & \end{bmatrix} = \Gamma$, let $C_k^\infty(\Gamma)$ be the space of smooth functions $F : \mathbb{H}_n \to \mathbb{C}$ satisfying

$$F(\gamma Z) = j(\gamma, Z)^k F(Z) \qquad \text{for all } \gamma \in \Gamma. \tag{108}$$

For any $F \in C_k^\infty(\Gamma)$, there is an element $\bar{F} \in C_k^\infty(\Gamma)$ defined via $\bar{F}(Z) = \overline{F(-\bar{Z})}$. Given functions F_1, F_2 in $C_k^\infty(\Gamma)$, we define the Petersson inner product $\langle F_1, F_2 \rangle$ by

$$\langle F_1, F_2 \rangle = \mathrm{vol}(\Gamma \backslash \mathbb{H}_n)^{-1} \int\limits_{\Gamma \backslash \mathbb{H}_n} \det(Y)^k F_1(Z) \overline{F_2(Z)} \, dZ,$$

whenever the integral above converges. Above, dZ is any $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant measure on $\mathbb{H}_n$ (it is equal to $c \det(Y)^{-(n+1)} \, dX \, dY$ for some constant c). Note that our definition of the Petersson inner product does not depend on the normalization of measure (the choice of c), and is also not affected by different choices of Γ. Note also that

$$\langle F_1, F_2 \rangle = \overline{\langle \bar{F}_1, \bar{F}_2 \rangle}. \tag{109}$$

Now, let Φ be an automorphic form on $G_{2n}(\mathbb{A})$ such that $\Phi(gh) = j(h, I)^{-k}\Phi(g)$ for all $h \in K_\infty^{(n)} \cong U(n)$. Then we can define a function $F(Z)$ on the Siegel upper half space $\mathbb{H}_n$ by

$$F(Z) = j(g, I)^k \, \Phi(g), \tag{110}$$

where g is any element of $\mathrm{Sp}_{2n}(\mathbb{R})$ with $g(I) = Z$. If Γ_p is an open-compact subgroup of $G_{2n}(\mathbb{Q}_p)$ such that Φ is right invariant under Γ_p for all p, then it is easy to check that $F \in C_k^\infty(\Gamma)$ where $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Q}) \cap \prod_{p<\infty} \Gamma_p$.

We apply this principle to our Eisenstein series $E(g, s, f)$, where f is the global section constructed in Sect. 6.1. Consider

$$E_{k,N}^\chi(Z, s) := j(g, I)^k E\Big(g, \frac{2s}{2n+1} + \frac{k}{2n+1} - \frac{1}{2}, f\Big), \tag{111}$$

where g is any element of $\mathrm{Sp}_{4n}(\mathbb{R})$ with $g(I) = Z$. Since the series defining $E(g, s, f)$ converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}$, it follows that the series defining $E_{k,N}^\chi(Z, s)$ converges absolutely whenever $2\mathrm{Re}(s) + k > 2n + 1$. More generally, for any $h \in \prod_{p<\infty} \mathrm{Sp}_{4n}(\mathbb{Z}_p)$ and g as above, define

$$E_{k,N}^\chi(Z, s; h) = j(g, I)^k E\Big(g, \frac{2s+k}{2n+1} - \frac{1}{2}, f^{(h)}\Big) = j(g, I)^k E\Big(gh^{-1}, \frac{2s+k}{2n+1} - \frac{1}{2}, f\Big). \tag{112}$$

By the invariance properties of our local sections and by analytic continuation, it follows that for all s and all h as above, we have $E_{k,N}^\chi(Z, s; h) \in C_k^\infty(\Gamma_{4n}(N))$.

It is instructive to write down the functions $E_{k,N}^\chi(Z, s; h)$ classically. First of all, a standard calculation shows that

$$E_{k,N}^\chi(Z, s; h) = \Big(E_{k,N}^\chi\big|_k h_0\Big)(Z, s) := j(h_0, Z)^{-k} E_{k,N}^\chi(h_0 Z, s), \tag{113}$$

where (using strong approximation) we let h_0 be an element of $\mathrm{Sp}_{4n}(\mathbb{Z})$ such that $h_0^{-1}h \in \prod_{p|N}\Gamma_{4n}(p^{m_p})\prod_{p\nmid N}\mathrm{Sp}_{4n}(\mathbb{Z}_p)$. This enables us to reduce to the case $h = 1$ for many properties. We now write down the classical definition in this case, i.e., for $E_{k,N}^\chi(Z,s)$. Let $P'_{4n}(\mathbb{Z}) = P_{4n}(\mathbb{Q}) \cap \mathrm{Sp}_{4n}(\mathbb{Z})$. Using the relevant definitions and the explication of the local sections $f_p(\gamma_p, s)$ at the end of Section 4.2, it follows that for $2\mathrm{Re}(s) + k > 2n + 1$,

$$E_{k,N}^\chi(Z,s) = \sum_{\gamma=\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]\in P'_{4n}(\mathbb{Z})\backslash\Gamma_{0,4n}(N)} \left(\prod_{p|N}\chi_p(\det(A))\right)\det(\mathrm{Im}(\gamma Z))^s\, j(\gamma, Z)^{-k}. \tag{114}$$

6.2 Remark. *Shimura defined certain Eisenstein series on symplectic and unitary groups over number fields [46, 47, 48, 49] and proved various properties about them. Following the notation of [49, Sect. 17], we denote Shimura's Eisenstein series by $E(z, s; k, \chi, N)$. A comparison of [49, (16.40)] and (114) shows that*

$$E_{k,N}^\chi(Z,s) = E(z, s + k/2; k, \chi, N). \tag{115}$$

The above identity can also be proved adelically, by comparing the alternate description of our section at ramified places (see end of Section 4.2) with Shimura's section (see Sect. 16.5 of [49]).

Combining (114) with (113), we can now write down a similar expansion for $E_{k,N}^\chi(Z,s;h)$ for each $h \in \prod_{p<\infty}\mathrm{Sp}_{4n}(\mathbb{Z}_p)$. Note that if Q is the element defined immediately above Theorem 6.1, then $E_{k,N}^\chi(Z,s;Q) = j(Q_n,Z)^{-k}E_{k,N}^\chi(Q_nZ,s)$.

Next, let $F(Z)$ be the function on $\mathbb{H}_n$ corresponding to the automorphic form ϕ via (110). Then $F, \bar{F} \in C_k^\infty(\Gamma_{2n}(N))$ and both these functions are rapidly decreasing at infinity (as ϕ is a cusp form). For $Z_1, Z_2 \in \mathbb{H}_n$, and $h \in \prod_{p<\infty}\mathrm{Sp}_{4n}(Z_p)$, write $E_{k,N}^\chi(Z_1, Z_2, s; h)$ for $E_{k,N}^\chi\left(\left[\begin{smallmatrix}Z_1 & \\ & Z_2\end{smallmatrix}\right], s; h\right)$. Using adelic-to-classical arguments similar to Theorem 6.5.1 of [38], we can now write down the classical analogue of Theorem 6.1.

6.3 Theorem. *Let the element Q_τ for each $\tau \in \hat{\mathbb{Z}}^\times$ be as defined just before Theorem 6.1, and let $E_{k,N}^\chi(Z, s; Q_\tau)$ be as defined in (112). Let $F(Z)$, $\bar{F}(Z)$ be as defined above. Then we have the relation*

$$\left\langle E_{k,N}^\chi\left(-, Z_2, \frac{n}{2} - \frac{k-s}{2}; Q_\tau\right), \bar{F}\right\rangle = \frac{L^N(s, \pi \boxtimes \chi, \varrho_{2n+1})}{L^N(s+n,\chi)\prod_{j=1}^n L^N(2s+2j-2,\chi^2)} \times A_{\mathbf{k}}(s-1)$$

$$\times \prod_{p|N}\mathrm{vol}(\Gamma_{2n}(p^{m_p})) \times \frac{\chi(\tau)^{-n}i^{nk}\pi^{n(n+1)/2}}{\mathrm{vol}(\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathrm{Sp}_{2n}(\mathbb{R}))} \times F(Z_2).$$

6.4 Corollary. *Let r be a positive integer satisfying $1 \le r \le k_n - n$, $r \equiv k_n - n \pmod{2}$. Then*

$$\left\langle E_{k,N}^\chi\left(-, Z_2, \frac{n}{2} - \frac{k-r}{2}; Q_\tau\right), \bar{F}\right\rangle = \frac{i^{nk}\chi(\tau)^{-n}c_{k,r,n,N}}{L^N(r+n,\chi)\prod_{j=1}^n L^N(2r+2j-2,\chi^2)}\frac{L^N(r, \pi \boxtimes \chi, \varrho_{2n+1})}{}F(Z_2)$$

where $c_{k,r,n,N} = \frac{\pi^{n(n+1)/2}\prod_{p|N}\mathrm{vol}(\Gamma_{2n}(p^{m_p}))}{\mathrm{vol}(\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathrm{Sp}_{2n}(\mathbb{R}))} \times A_{\mathbf{k}}(r-1)$ is a non-zero rational number.

6.3   Near holomorphy and rationality of Eisenstein series

In this subsection, we will prove two important properties of the Eisenstein series $E^\chi_{k,N}(Z, -m; h)$ for certain non-negative integers m. These are stated as Propositions 6.5 and 6.7.

For each positive integer r, let $N(\mathbb{H}_r)$ be the space of nearly holomorphic functions on $\mathbb{H}_r$. By definition, these are the functions which are polynomials in the entries of $\text{Im}(Z)^{-1}$ with holomorphic functions on $\mathbb{H}_r$ as coefficients. For each discrete subgroup Γ of $\text{Sp}_{2r}(\mathbb{R})$, let $N_k(\Gamma)$ be the space of all functions F in $N(\mathbb{H}_r)$ that satisfy $F(\gamma Z) = j(\gamma, Z)^k F(Z)$ for all $Z \in \mathbb{H}_r$ and $\gamma \in \Gamma$ (if $r = 1$, we also need an additional "no poles at cusps" condition, as explained in [35]). The spaces $N_k(\Gamma)$ are known as nearly holomorphic modular forms of weight k for Γ. We let $M_k(\Gamma) \subset N_k(\Gamma)$ denote the usual space of holomorphic modular forms of weight k for Γ. The following important result is due to Shimura.

6.5 Proposition. *Suppose that $k \geq n + 1$ and let $h \in \prod_{p<\infty} \text{Sp}_4(\mathbb{Z}_p)$. If $k = n + 1$, assume further that $\chi^2 \neq 1$. Then $E^\chi_{k,N}(Z, 0; h)$ is a holomorphic Siegel modular form of degree $2n$ and weight k with respect to the principal congruence subgroup $\Gamma_{4n}(N)$ of $\text{Sp}_{4n}(\mathbb{R})$. More generally, let $0 \leq m \leq \frac{k}{2} - \frac{n+1}{2}$ be an integer, and exclude the case $m = \frac{k}{2} - \frac{n+1}{2}$, $\chi^2 = 1$. Then*

$$E^\chi_{k,N}(Z, -m; h) \in N_k(\Gamma_{4n}(N)).$$

Proof. If $h = 1$, this is a special case of Theorem 17.9 of [49] (see Remark 6.2). The proof for general h is now an immediate consequence of (113). ∎

6.6 Remark. *In the absolutely convergent range $k \geq 2n + 2$, $0 \leq m \leq \frac{k}{2} - n - 1$, the above Proposition can also be proved directly, using the expansion (114). To go beyond the realm of absolute convergence, one needs delicate results involving analytic behavior of Fourier coefficients of Eisenstein series, which have been done by Shimura.*

Next, for any nearly holomorphic modular form $F \in N_k(\Gamma)$, we let $^\sigma F$ denote the nearly holomorphic modular form obtained by letting σ act on the Fourier coefficients of F. Note that if σ is complex conjugation then $^\sigma F$ equals $\bar{F}$. Denote

$$h_\tau := \begin{bmatrix} \tau I_{2n} & \\ & I_{2n} \end{bmatrix} h \begin{bmatrix} \tau^{-1} I_{2n} & \\ & I_{2n} \end{bmatrix}.$$

We will prove the following result.

6.7 Proposition. *Let the setup be as in Proposition 6.5. Let $\sigma \in \text{Aut}(\mathbb{C})$, and let $\tau \in \hat{\mathbb{Z}}^\times$ be the element corresponding to σ via the natural map $\text{Aut}(\mathbb{C}) \to \text{Gal}(\mathbb{Q}_{\text{ab}}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^\times$. (Concretely, this means that for each positive integer m, $\sigma(e^{2\pi i/m}) = e^{2\pi it/m}$ where $t \equiv \tau \mod m$.) Then we have*

$$^\sigma E^\chi_{k,N}(Z, -m; h) = \frac{\sigma(\pi^{2mn})}{\pi^{2mn}} E^{\sigma\chi}_{k,N}(Z, -m; h_\tau).$$

Proof. We will prove the result in several steps.

6.8 Lemma. (Feit) *Assume that $k \geq n + 1$, and exclude the case $k = n + 1, \chi^2 = 1$. Then $^\sigma E^\chi_{k,N}(Z, 0) = E^{\sigma\chi}_{k,N}(Z, 0)$. In other words, Proposition 6.7 holds for $m = 0$, $h = 1$.*

Proof. This is Theorem 15.1 of [13].   ■

The next step is to extend the above lemma to the case of general h. This follows from a very general lemma of Shimura. For any Siegel modular form $F \in M_k(\Gamma_{2r}(N))$ and any $h \in \prod_{p<\infty} \mathrm{Sp}_{2r}(\mathbb{Z}_p)$, define $F|_k h \in M_k(\Gamma_{2r}(N))$ by $F|_k h := F|_k h_0$ where we choose h_0 to be any element of $\mathrm{Sp}_{2r}(\mathbb{Z})$ with $(h_0^{-1}h)_p \in \Gamma_{2r}(N)$ for all p. Note that $E_{k,N}^\chi(Z,0)|_k h = E_{k,N}^\chi(Z,0;h)$.

6.9 Lemma. (Shimura) *Let σ, τ be as in Proposition* 6.7. *Then, for all $h \in \prod_{p<\infty} \mathrm{Sp}_{2r}(\mathbb{Z}_p)$, and $F \in M_k(\Gamma_{2r}(N))$, we have $^\sigma(F|_k h) = (^\sigma F)|_k h_\tau$.*

Proof. This is immediate from Lemma 10.5 of [49] and its proof.   ■

Combining the above two lemmas, since $k - 2m \geq n+1$, we see now that

$$^\sigma E_{k-2m,N}^\chi(Z,0;h) = E_{k-2m,N}^{\sigma\chi}(Z,0;h_\tau) \tag{116}$$

for all k, m, χ as in Proposition 6.5. Next, we need the Maass-Shimura differential operator Δ_k^p which is defined in [47, 4.10a] or [49, p. 146]. Note that Δ_k^p takes $N_k(\Gamma)$ to $N_{k+2p}(\Gamma)$. By [49, (17.21)] we obtain

$$\Delta_{k-2m}^m(E_{k-2m,N}^\chi(Z,0;h)) = dE_{k,N}^\chi(Z,-m;h) \tag{117}$$

where d is a non-zero rational number. (Note here that the differential operator Δ_k^p *commutes* with the action of h). Finally, we have the identity (see equation (1) of [10]):

$$^\sigma(\pi^{-2mn}\Delta_{k-2m}^m F) = \Delta_{k-2m}^m(\pi^{-2mn} \, {}^\sigma F). \tag{118}$$

Combining (116), (117), and (118), we conclude the proof of Proposition 6.7.   ■

6.10 Remark. *In the special case $m = 0$, $k \geq 2n+2$, where the Eisenstein series is holomorphic and absolutely convergent, Proposition* 6.7 *also appears in* [18, p. 461].

7 Critical L-values for $\mathrm{GSp}_4 \times \mathrm{GL}_1$

7.1 Preliminaries

Throughout this section we assume $n = 2$. Let ℓ, m be non-negative integers with m even and $\ell \geq 3$. We put $\mathbf{k} = (\ell+m)e_1 + \ell e_2$ and $k = \ell + m$. For each integer $N = \prod_p p^{m_p}$, we let $\Pi_N(\mathbf{k})$ denote the set of all cuspidal automorphic representations $\pi \cong \otimes \pi_p$ of $G_4(\mathbb{A})$ such that π_∞ equals the holomorphic discrete series representation $\pi_\mathbf{k}$ and such that for each finite prime p, π_p has a vector right invariant under the principal congruence subgroup $\Gamma_4(p^{m_p})$ of $\mathrm{Sp}_4(\mathbb{Z}_p)$. We put $\Pi(\mathbf{k}) = \bigcup_N \Pi_N(\mathbf{k})$.

We say that a character $\chi = \otimes \chi_p$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ is a *Dirichlet character* if χ_∞ is trivial on $\mathbb{R}_{>0}$. Any such χ gives rise to a homomorphism $\tilde\chi : (\mathbb{Z}/N_\chi\mathbb{Z})^\times \to \mathbb{C}^\times$, where N_χ denotes the conductor of χ. Concretely[5] the map $\tilde\chi$ is given by $\tilde\chi(a) = \prod_{p|N_\chi} \chi_p^{-1}(a)$. Given a Dirichlet character χ, we define the corresponding Gauss sum by $G(\chi) = \sum_{n\in(\mathbb{Z}/N_\chi\mathbb{Z})^\times} \tilde\chi(n)e^{2\pi in/N_\chi}$.

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[5]In fact, the map $\chi \mapsto \tilde\chi$ gives a bijection between Dirichlet characters in our sense and primitive Dirichlet characters in the classical sense.

7.1 Lemma. *Let χ, χ' be Dirichlet characters. Given $\sigma \in \mathrm{Aut}(\mathbb{C})$, let $\tau \in \hat{\mathbb{Z}}^\times$ be as in Proposition 6.7. The following hold:*

i) $\sigma(G(\chi)) = \chi(\tau)G(^\sigma\chi)$.

ii) $\sigma\left(\frac{G(\chi\chi')}{G(\chi)G(\chi')}\right) = \frac{G(^\sigma\chi\,^\sigma\chi')}{G(^\sigma\chi)G(^\sigma\chi')}$.

Proof. This is well-known; see, e.g., [10, Lemma 4]. We omit the proof, which is elementary. ∎

Let $\mathcal{X}_N$ be the set of all Dirichlet characters χ such that $\chi_\infty = \mathrm{sgn}^\ell$ and $N_\chi|N$. We denote $\mathcal{X} = \bigcup_N \mathcal{X}_N$. For any $\pi \in \Pi_N(\mathbf{k})$, $\chi \in \mathcal{X}_N$, and any positive integer r, we define, following the notation of Corollary 6.4,

$$C_N(\pi, \chi, r) := (-1)^k \pi^{2r+4-2k} \ c_{k,r,2,N} \ \frac{L^N(r, \pi \boxtimes \chi, \varrho_5)}{L^N(r+2, \chi)L^N(2r, \chi^2)L^N(2r+2, \chi^2)}.$$

The reader should not be confused by the two different π (one the constant, the other an automorphic representation) appearing in the above definition. For the rest of this paper, we also make the following assumption (which is forced upon us as we need to use results of Shimura where this assumption appears):

$$r = 1 \implies \chi^2 \neq 1 \tag{119}$$

The reader might wonder if $C_N(\pi, \chi, r)$ can be infinite. It turns out that (119) eliminates that possibility. First of all, any $\pi \in \Pi_N(\mathbf{k})$ is either of generic type (meaning, it lifts to a cusp form on GL_4), or of endoscopic (Yoshida) type, or of P-CAP (Saito-Kurokawa) type. (CAP representations of Soudry type or Howe–Piatetski-Shapiro type do not occur if $\ell \geq 3$.) In each case, we have precise information about the possible poles of $L(r, \pi \boxtimes \chi, \varrho_5)$; see [40]. It follows that if π is generic, then $L^N(r, \pi \boxtimes \chi, \varrho_5)$ is finite for all $r \geq 1$. On the other hand, for π either endoscopic or P-CAP, $L^N(r, \pi \boxtimes \chi, \varrho_5)$ is finite for $r > 1$ and $L^N(r, \pi \boxtimes \chi, \varrho_5) = \infty \Rightarrow \chi = 1$. In particular, assumption (119) implies that $C_N(\pi, \chi, r)$ is finite in all cases considered by us.

Recall that $N_k(\Gamma_4(N))$ denotes the (finite-dimensional) space of nearly holomorphic modular forms of weight k for the subgroup $\Gamma_4(N)$ of $\mathrm{Sp}_4(\mathbb{Z})$. Let V_N be the subset of $N_k(\Gamma_4(N))$ consisting of those forms F which are *cuspidal* and for which the corresponding function Φ_F on $\mathrm{Sp}_4(\mathbb{R})$ generates an irreducible representation isomorphic to $\pi_{\mathbf{k}}$. By Theorem 4.8 and Proposition 4.28 of [35], V_N is a *subspace* of $N_k(\Gamma_4(N))$ and isomorphic to the space $S_{\ell,m}(\Gamma_4(N))$ of holomorphic vector-valued cusp forms of weight $\det^\ell \mathrm{sym}^m$ for $\Gamma_4(N)$; indeed $V_N = U^{m/2}(S_{\ell,m}(\Gamma_4(N)))$, where U is the differential operator defined in Section 3.4 of [35]. We put $V = \bigcup_N V_N$ and $N_k = \bigcup_N N_k(\Gamma_4(N))$.

As in [35], we let $\mathfrak{p}^\circ_{\ell,m}$ denote the orthogonal projection map from N_k to V; note that it takes $N_k(\Gamma_4(N))$ to V_N for each N. Let $1 \leq r \leq \ell - 2$, $r \equiv \ell \pmod 2$ be an integer and $\chi \in \mathcal{X}_N$; also suppose that (119) holds. Then $E^\chi_{k,N}(Z_1, Z_2, 1 - \frac{k-r}{2}; Q_\tau) \in N_k(\Gamma_4(N)) \otimes N_k(\Gamma_4(N))$ for each $\tau \in \hat{\mathbb{Z}}^\times$. In fact, it can be shown that $E^\chi_{k,N}(Z_1, Z_2, 1 - \frac{k-r}{2}; Q_\tau)$ is cuspidal in each variable using methods very similar to [18], but we will not need this. We define

$$G^\chi_{k,N}(Z_1, Z_2, r; Q_\tau) := \pi^{2r+4-2k} \times (\mathfrak{p}^\circ_{\ell,m} \otimes \mathfrak{p}^\circ_{\ell,m})(E^\chi_{k,N}(Z_1, Z_2, 1 - \frac{k-r}{2}; Q_\tau)). \tag{120}$$

If $F \in V$ is such that (the adelization of) F generates a multiple of an irreducible (cuspidal, automorphic) representation of $G_4(\mathbb{A})$, then we let π_F denote the representation associated to F. Note that the set of automorphic representations π_F obtained this way as F varies in V_N is precisely equal to the set $\Pi_N(\mathbf{k})$ defined above. For each $\pi \in \Pi(\mathbf{k})$, we let $V_N(\pi)$ denote the π-isotypic part of V_N. Precisely, this is the subspace consisting of all those F in V_N such that all irreducible constituents of the representation generated by (the adelization of) F are isomorphic to π. Note that $V_N(\pi) = \{0\}$ unless $\pi \in \Pi_N(\mathbf{k})$. We have an orthogonal direct sum decomposition

$$V_N = \bigoplus_{\pi \in \Pi_N(\mathbf{k})} V_N(\pi). \tag{121}$$

We define $V(\pi) = \bigcup_N V_N(\pi)$. Therefore we have $V = \bigoplus_{\pi \in \Pi(\mathbf{k})} V(\pi)$. Now, let $\mathfrak{B}$ be *any* orthogonal basis of V_N formed by taking a union of orthogonal bases from the right side of (121). Thus each $F \in \mathfrak{B}$ belongs to $V_N(\pi)$ for some $\pi \in \Pi_N(\mathbf{k})$. From Corollary 6.4, Proposition 6.5, and (109), we deduce the following key identity:

$$G^\chi_{k,N}(Z_1, Z_2, r; Q_\tau) = \chi(\tau)^{-2} \sum_{F \in \mathfrak{B}} C_N(\pi_F, \chi, r) \frac{F(Z_1)\bar{F}(Z_2)}{\langle F, F \rangle}$$

$$= \chi(\tau)^{-2} \sum_{\substack{\pi \in \Pi_N(\mathbf{k})}} C_N(\pi, \chi, r) \sum_{\substack{F \in \mathfrak{B} \\ F \in V_N(\pi)}} \frac{F(Z_1)\bar{F}(Z_2)}{\langle F, F \rangle}. \tag{122}$$

7.2   Arithmeticity

Throughout this section, we assume that $1 \leq r \leq \ell - 2$, $r \equiv \ell \pmod 2$, and that (119) holds.

7.2 Lemma. *For all* $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$^\sigma G^\chi_{k,N}(Z_1, Z_2, r; Q) = G^{\sigma\chi}_{k,N}(Z_1, Z_2, r; Q_\tau).$$

Proof. Recall that $G^\chi_{k,N}(Z_1, Z_2, r; Q_\tau)$ is defined by (120). So the corollary follows from Proposition 6.7 (taking $h = Q$) and the fact that the map $\mathfrak{p}^\circ_{\ell,m}$ commutes with the action of $\mathrm{Aut}(\mathbb{C})$ (see Proposition 5.17 of [35]). ∎

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. For $\pi \in \Pi_N(\mathbf{k})$, we let $^\sigma\pi \in \Pi_N(\mathbf{k})$ be the representation obtained by the action of σ, and we let $\mathbb{Q}(\pi)$ denote the field of rationality of π; see the beginning of Section 3.4 of [39]. If σ is the complex conjugation, then we denote $^\sigma\pi = \bar{\pi}$. It is known that $\mathbb{Q}(\pi)$ is a CM field and $\mathbb{Q}(\bar{\pi}) = \mathbb{Q}(\pi)$. We use $\mathbb{Q}(\bar{\pi}, \chi)$ to denote the compositum of $\mathbb{Q}(\pi)$ and $\mathbb{Q}(\chi)$. Note that

$$F \in V_N(\pi) \implies {}^\sigma F \in V_N({}^\sigma\pi). \tag{123}$$

This follows from Theorem 4.2.3 of [4] (see the proof of Proposition 3.13 of [39]) together with the fact that the U operator commutes with σ. In particular, the space $V_N(\pi)$ is preserved under the action of the group $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi))$. Using Lemma 3.17 of [39], it follows that the space $V_N(\pi)$ has a basis consisting of forms whose Fourier coefficients are in $\mathbb{Q}(\pi)$. In particular there exists some F satisfying the conditions of the next proposition.

7.3 Proposition. *Let $\pi \in \Pi_N(\mathbf{k})$, and $F \in V_N(\pi)$. Suppose that the Fourier coefficients of F lie in a CM field. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\frac{G(\chi^2)C_N(\pi,\chi,r)}{\langle F,F\rangle}\right) = \frac{G(^\sigma\chi^2)C_N(^\sigma\pi,{}^\sigma\chi,r)}{\langle {}^\sigma F,{}^\sigma F\rangle}.$$

Proof. Let us complete F to an orthogonal basis $\mathfrak{B} = \{F = F_1, F_2, \ldots, F_r\}$ of $V_N(\pi)$. Let $\mathfrak{B}' = \{G_1, \ldots, G_r\}$ be any orthogonal basis for $V_N(^\sigma\pi)$. Given σ, let τ be as in Proposition 6.7. Using (122), (123), and Lemma 7.2, and comparing the $V_N(^\sigma\pi)$ components, we see that

$$\sigma(C_N(\pi,\chi,r)) \sum_i \frac{^\sigma F_i(Z_1)^\sigma\bar{F}_i(Z_2)}{\sigma(\langle F_i,F_i\rangle)} = \chi^{-2}(\tau)C_N(^\sigma\pi,{}^\sigma\chi,r)\sum_i \frac{G_i(Z_1)\bar{G}_i(Z_2)}{\langle G_i,G_i\rangle}.$$

Taking inner products of each side with $^\sigma F_1$ (in the variable Z_1) we deduce that

$$\sigma(C_N(\pi,\chi,r))\sum_i \frac{\langle {}^\sigma F_i, {}^\sigma F_1\rangle}{\sigma(\langle F_i,F_i\rangle)}{}^\sigma \bar{F}_i(Z_2) = \chi^{-2}(\tau)C_N(^\sigma\pi,{}^\sigma\chi,r)\overline{{}^\sigma F_1}(Z_2).$$

Noting that $\overline{{}^\sigma F_1} = {}^\sigma\bar{F}_1$ and comparing the coefficients of $^\sigma\bar{F}_1(Z_2)$ on each side, we conclude the desired equality, using Lemma 7.1. ∎

7.3   The main result on critical L-values

For each $p|N$, we define the local L-factor $L(s,\pi_p\boxtimes\chi_p,\varrho_5)$ via the local Langlands correspondence [15]. In particular, $L(s,\pi_p \boxtimes \chi_p,\varrho_5)$ is just a local L-factor for $\mathrm{GL}_5 \times \mathrm{GL}_1$. This definition also works at the good places, and indeed coincides with what we previously defined. For any finite set of places S of $\mathbb{Q}$, we define the global L-function

$$L^S(r,\pi\boxtimes\chi,\varrho_5) = \prod_{p\notin S} L(s,\pi_p\boxtimes\chi_p,\varrho_5).$$

7.4 Theorem. *Let $\pi \in \Pi(\mathbf{k})$, and let $F \in V(\pi)$ be such that its Fourier coefficients lie in a CM field. Let S be any finite set of places of $\mathbb{Q}$ containing the infinite place, $\chi \in \mathcal{X}$, r be an integer satisfying $1 \le r \le \ell - 2$, $r \equiv \ell \pmod 2$. If $r = 1$, assume $\chi^2 \ne 1$. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have*

$$\sigma\left(\frac{L^S(r,\pi\boxtimes\chi,\varrho_5)}{(2\pi i)^{2k+3r}G(\chi)^3\langle F,F\rangle}\right) = \frac{L^S(r,{}^\sigma\pi\boxtimes{}^\sigma\chi,\varrho_5)}{(2\pi i)^{2k+3r}G(^\sigma\chi)^3\langle {}^\sigma F,{}^\sigma F\rangle}. \tag{124}$$

Proof. Let $\chi \in \mathcal{X}$. We choose some N such that $F \in V_N(\pi)$ and $\chi \in \mathcal{X}_N$. By Proposition 7.3,

$$\sigma\left(\frac{G(\chi^2)L^N(r,\pi\boxtimes\chi,\varrho_5)}{\pi^{2k-2r-4}L^N(r+2,\chi)L^N(2r,\chi^2)L^N(2r+2,\chi^2)\langle F,F\rangle}\right)$$
$$= \frac{G(^\sigma\chi^2)L^N(r,{}^\sigma\pi\boxtimes{}^\sigma\chi,\varrho_5)}{\pi^{2k-2r-4}L^N(r+2,{}^\sigma\chi)L^N(2r,{}^\sigma\chi^2)L^N(2r+2,{}^\sigma\chi^2)\langle {}^\sigma F,{}^\sigma F\rangle}. \tag{125}$$

Next, for any finite prime p such that $p \in S$, $p|N$, and any Dirichlet character ψ, it follows from Lemma 4.6 of [11] that

$$\sigma(L(r,\pi_p\boxtimes\chi_p,\varrho_5)) = L(r,{}^\sigma\pi_p\boxtimes{}^\sigma\chi_p,\varrho_5), \quad \sigma(L(r,\psi_p)) = L(r,{}^\sigma\psi_p). \tag{126}$$

Finally, for any Dirichlet character ψ, we have the following fact (see [44, Lemma 5]) for any positive integer t satisfying $\psi(-1) = (-1)^t$:

$$\sigma\left(\frac{L^N(t, \psi)}{(2\pi i)^t G(\psi)}\right) = \frac{L^N(t, {}^\sigma\psi)}{(2\pi i)^t G({}^\sigma\psi)}. \tag{127}$$

Combining (125), (126), (127) (with $\psi = \chi$ and χ^2) we get the desired result. ∎

7.5 Remark. *Let F be as in the above theorem. By the results of [35], we know that $F = U^{m/2} F_0$ where F_0 is a holomorphic vector-valued Siegel cusp form. Using Lemma 4.16 of [35], we have moreover the equality $\langle F_0, F_0 \rangle = c_{\ell,m} \langle F, F \rangle$ for some constant $c_{\ell,m}$ that depends only on ℓ and m. By restricting to the special case of a full-level vector valued Siegel cusp form of weight $\det^\ell \mathrm{sym}^m$, and comparing Theorem 7.4 with the result of [29], we see that $c_{\ell,m}$ is a rational multiple of π^m. Hence in the theorem above, the term $\langle F, F \rangle$ can be replaced by $\pi^{-m} \langle F_0, F_0 \rangle$.*

7.6 Definition. *For two representations π_1, π_2 in $\Pi(\mathbf{k})$, we write $\pi_1 \approx \pi_2$ if there is a Hecke character ψ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ such that π_1 is nearly equivalent to $\pi_2 \otimes \psi$.*

Note that if such a ψ as above exists, then ψ_∞ must be trivial on $\mathbb{R}_{>0}$ and therefore ψ must be a Dirichlet character. The relation $\approx$ clearly gives an equivalence relation on $\Pi(\mathbf{k})$. For any $\pi \in \Pi(\mathbf{k})$, let $[\pi]$ denote the class of π, i.e., the set of all representations π_0 in $\Pi(\mathbf{k})$ satisfying $\pi_0 \approx \pi$. For any integer N, we define the subspace $V_N([\pi])$ of V_N to be the (direct) sum of all the subspaces $V_N(\pi_0)$ where π_0 ranges over all the inequivalent representations in $[\pi] \cap \Pi_N(\mathbf{k})$.

7.7 Corollary. *Let $\pi_1, \pi_2 \in \Pi(\mathbf{k})$ be such that $\pi_1 \approx \pi_2$. Let $F_1 \in V(\pi_1)$ and $F_2 \in V(\pi_2)$ have coefficients in a CM field. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have*

$$\sigma\left(\frac{\langle F_1, F_1 \rangle}{\langle F_2, F_2 \rangle}\right) = \frac{\langle {}^\sigma F_1, {}^\sigma F_1 \rangle}{\langle {}^\sigma F_2, {}^\sigma F_2 \rangle}.$$

Proof. By assumption, there is a character ψ and a set S of places containing the infinite place, such that $\pi_{1,p} \simeq \pi_{2,p} \otimes \psi_p$ for all $p \notin S$. We fix any character $\chi \in \mathcal{X}$. Note that $L(s, \pi_{1,p} \boxtimes \chi_p, \varrho_5) = L(s, \pi_{1,p} \boxtimes \chi_p, \varrho_5)$ for all $p \notin S$, as the representation ϱ_5 factors through PGSp_4 and therefore is blind to twisting by ψ. Applying Theorem 7.4 twice at the point $r = \ell - 2$, first with (π_1, F_1), and then with (π_2, F_2), and dividing the two equalities, we get the desired result. ∎

We can now prove Theorem 1.1. For each $\pi \in \Pi(\mathbf{k})$ we need to define a quantity $C(\pi)$ that has the properties claimed in Theorem 1.1. The first two properties required by Theorem 1.1 can be summarized as saying that $C(\pi)$ depends only on the class $[\pi]$ of π. For each $\pi \in \Pi(\mathbf{k})$, we let $N([\pi])$ denote the smallest integer such that $[\pi] \cap \Pi_{N([\pi])}(\mathbf{k}) \neq \emptyset$. Next, note that if $\pi_1 \approx \pi_2$, then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^\sigma \pi_1 \approx {}^\sigma \pi_2$. Thus for any class $[\pi]$ and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have a well-defined notion of the class ${}^\sigma[\pi] = [{}^\sigma \pi]$. Let $\mathbb{Q}([\pi])$ denote the fixed field of the set of all automorphisms of $\mathbb{C}$ such that ${}^\sigma[\pi] = [\pi]$. Clearly the field $\mathbb{Q}([\pi])$ is contained in $\mathbb{Q}(\pi_0)$ for any $\pi_0 \in [\pi]$. It is also clear that $N({}^\sigma[\pi]) = N([\pi])$ and $\mathbb{Q}({}^\sigma[\pi]) = \sigma(\mathbb{Q}([\pi]))$ for any $\sigma \in \mathrm{Aut}(\mathbb{C})$.

The space $V_{N([\pi])}([\pi])$ is preserved under the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}([\pi]))$ and therefore has a basis consisting of forms with coefficients in $\mathbb{Q}([\pi])$. Suppose that G is a non-zero element of

$V_{N([\pi])}([\pi])$ whose Fourier coefficients lie in $\mathbb{Q}([\pi])$. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, $^{\sigma}G$ is an element of $V_{N(^{\sigma}[\pi])}(^{\sigma}[\pi])$ whose Fourier coefficients lie in $\mathbb{Q}(^{\sigma}[\pi])$. Moreover, if σ, τ are elements of $\mathrm{Aut}(\mathbb{C})$ such that $^{\sigma}[\pi] = {}^{\tau}[\pi]$, then $^{\sigma}G = {}^{\tau}G$. It follows that for each class $[\pi]$, we can choose a non-zero element $G_{[\pi]} \in V_{N([\pi])}([\pi])$ such that the following two properties hold:

i) The Fourier coefficients of $G_{[\pi]}$ lie in $\mathbb{Q}([\pi])$.

ii) $G_{\sigma[\pi]} = {}^{\sigma}G_{[\pi]}$.

We do *not* require that the adelization of $G_{[\pi]}$ should generate an irreducible representation.

Finally, for any $\pi \in \Pi(\mathbf{k})$ we define

$$C(\pi) = (2\pi i)^{2k}\langle G_{[\pi]}, G_{[\pi]}\rangle.$$

By construction, $C(\pi)$ depends only on the class $[\pi]$ of π. So we only need to prove (1). The following lemma is key.

7.8 Lemma. *Let π, F be as in Theorem 7.4. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{\langle F, F\rangle}{\langle G_{[\pi]}, G_{[\pi]}\rangle}\right) = \frac{\langle {}^{\sigma}F, {}^{\sigma}F\rangle}{\langle {}^{\sigma}G_{[\pi]}, {}^{\sigma}G_{[\pi]}\rangle}.$$

Proof. Write $G_{[\pi]} = F_1 + F_2 + \ldots + F_t$ where $F_i \in V(\pi_i)$ for inequivalent representations $\pi_i \in [\pi]$. Note that the spaces $V(\pi_i)$ are mutually orthogonal and hence

$$\langle G_{[\pi]}, G_{[\pi]}\rangle = \sum_{i=1}^{t}\langle F_i, F_i\rangle, \quad \langle {}^{\sigma}G_{[\pi]}, {}^{\sigma}G_{[\pi]}\rangle = \sum_{i=1}^{t}\langle {}^{\sigma}F_i, {}^{\sigma}F_i\rangle.$$

So the desired result would follow immediately from Corollary 7.7 provided we can show that each F_i has coefficients in a CM field.

Indeed, let K be the compositum of all the fields $\mathbb{Q}(\pi_i)$. Thus K is a CM field containing $\mathbb{Q}([\pi])$. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, we have $^{\sigma}G_{[\pi]} = G_{[\pi]}$ and $^{\sigma}F_i \in V(\pi_i)$. As the spaces $V(\pi_i)$ are all linearly independent, it follows that $^{\sigma}F_i = F_i$ and therefore each F_i has coefficients in a CM field. ∎

The proof of (1) follows by combining Theorem 7.4 and Lemma 7.8.

7.4   Symmetric fourth L-function of GL_2

Let k be an even positive integer and M any positive integer. Let f be an elliptic cuspidal newform of weight k, level M and trivial nebentypus that is not of dihedral type. According to Theorem A' and Theorem C of [37], there exists a cuspidal automorphic representation π of $\mathrm{GSp}_4(\mathbb{A})$ such that

i) π_∞ is the holomorphic discrete series representation with highest weight $(2k - 1, k + 1)$,

ii) for $p \nmid M$, the local representation π_p is unramified,

iii) the L-functions have the following relation.

$$L(s, \pi, \varrho_5) = L(s, \mathrm{sym}^4 f).$$

Note that π corresponds to a holomorphic vector-valued Siegel cusp form F_0 with weight $\det^{k+1} \mathrm{sym}^{k-2}$. Let χ be a Dirichlet character in $\mathcal{X}$, i.e., $\chi_\infty = \mathrm{sgn}$. We have

$$L(s, \pi \boxtimes \chi, \varrho_5) = L(s, \chi \otimes \mathrm{sym}^4 f). \tag{128}$$

Here, on the right hand side, we have the L-function of GL_5 given by the symmetric fourth power of f, twisted by the character χ.

7.9 Theorem. *Let $F \in V(\pi)$ be such that its Fourier coefficients lie in a CM field. Let S be any finite set of places of $\mathbb{Q}$ containing the infinite place, $\chi \in \mathcal{X}$, and r be an integer satisfying $1 \le r \le k-1$, r odd. If $r = 1$, assume $\chi^2 \ne 1$. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have*

$$\sigma \left(\frac{L^S(r, \chi \otimes \mathrm{sym}^4 f)}{(2\pi i)^{4k-2+3r} G(\chi)^3 \langle F, F \rangle} \right) = \frac{L^S(r, {}^\sigma\chi \otimes \mathrm{sym}^4({}^\sigma f))}{(2\pi i)^{4k-2+3r} G({}^\sigma\chi)^3 \langle {}^\sigma F, {}^\sigma F \rangle}. \tag{129}$$

Proof. This theorem follows from Theorem 7.4 and (128). ∎

Proof of Theorem 1.5. This follows similarly, only using Theorem 1.1 rather than Theorem 7.4.

7.10 Remark. *Deligne's famous conjecture on critical values of motivic L-functions predicts an algebraicity result for the critical values of $L(s, \chi \otimes \mathrm{sym}^m f)$ for each positive integer m. For $m = 1$ this was proved by Shimura [45], for $m = 2$ by Sturm [50], and for $m = 3$ by Garrett–Harris [16]. Assuming functoriality, this was also proved for all odd m by Raghuram [36]. To the best of our knowledge, the results of this paper represent the first advances in the case $m = 4$.*

A   Haar measures on $\mathrm{Sp}_{2n}(\mathbb{R})$

This appendix will furnish proofs for the constants appearing in the integration formulas (27) and (85). The symbol K denotes the maximal compact subgroup $\mathrm{Sp}_{2n}(\mathbb{R}) \cap \mathrm{O}(2n)$ of $\mathrm{Sp}_{2n}(\mathbb{R})$.

A.1   The KAK measure

Recall that we have fixed the "classical" Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$ characterized by the property (12). There is also the "KAK measure" given by the integral on the right hand side of (27).

A.1 Proposition. *The constant α_n in (27) is given by*

$$\alpha_n = \frac{(4\pi)^{n(n+1)/2}}{\prod_{m=1}^{n}(m-1)!}. \tag{130}$$

Proof. The proof consists of evaluating both sides of (27) for the function

$$F(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = \frac{2^{2nk}}{|\det(A+D+i(C-B))|^{2k}}. \tag{131}$$

Note that this function is the square of the absolute value of the matrix coefficient appearing in (65). Hence the integrals will be convergent as long as $k > n$. The function $f(Z)$ on $\mathbb{H}_n$ corresponding to F is given by

$$
\begin{aligned}
f(Z) &= F(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\begin{bmatrix} Y^{1/2} & \\ & Y^{-1/2} \end{bmatrix}) \\
&= \frac{2^{2nk}}{|\det(Y^{1/2} + Y^{-1/2} - iXY^{-1/2})|^{2k}} \\
&= \frac{2^{2nk}\det(Y)^k}{|\det(1_n + Y - iX)|^{2k}}.
\end{aligned}
\tag{132}
$$

Hence, by (12),

$$
\int\limits_{\mathrm{Sp}_{2n}(\mathbb{R})} F(g)\,dg = 2^{2nk}\int\limits_{\mathbb{H}_n}\frac{\det(Y)^{k-n-1}}{|\det(1_n + Y - iX)|^{2k}}\,dX\,dY.
\tag{133}
$$

We now employ the following integral formula. For a matrix X, denote by $[X]_p$ the upper left block of size $p \times p$ of X. For $j = 1,\ldots,n$ let $\lambda_j, \sigma_j, \tau_j$ be complex numbers, and set $\lambda_{n+1} = \sigma_{n+1} = \tau_{n+1} = 0$. Then, by (0.11) of [33],

$$
\begin{aligned}
&\int\limits_{\mathbb{H}_n}\prod_{j=1}^{n}\frac{\det[Y]_j^{\lambda_j - \lambda_{j+1}}}{\det[1_n + Y + iX]_j^{\sigma_j - \sigma_{j+1}}\det[1_n + Y - iX]_j^{\tau_j - \tau_{j+1}}}\det(Y)^{-(n+1)}\,dX\,dY \\
&= \prod_{m=1}^{n}\frac{2^{2-\sigma_m - \tau_m + n - m}\,\pi^m\,\Gamma(\lambda_m - (n+m)/2)\,\Gamma(\sigma_m + \tau_m - \lambda_m - (n-m)/2)}{\Gamma(\sigma_m - (n-m)/2)\,\Gamma(\tau_m - (n-m)/2)},
\end{aligned}
\tag{134}
$$

provided the integral is convergent. We will only need the special case where all λ_j are equal to some λ, all σ_j are equal to some σ, and all τ_j are equal to some τ. In this case the formula says that

$$
\begin{aligned}
&\int\limits_{\mathbb{H}_n}\frac{\det(Y)^\lambda}{\det(1_n + Y + iX)^\sigma\det(1_n + Y - iX)^\tau}\det(Y)^{-(n+1)}\,dX\,dY \\
&= 2^{-(\sigma+\tau)n + n(n+3)/2}\pi^{n(n+1)/2}\prod_{m=1}^{n}\frac{\Gamma(\lambda - (n+m)/2)\Gamma(\sigma + \tau - \lambda - (n-m)/2)}{\Gamma(\sigma - (n-m)/2)\Gamma(\tau - (n-m)/2)}.
\end{aligned}
\tag{135}
$$

If we set $\lambda = \sigma = \tau = k$, we obtain precisely the integral (133). Hence,

$$
\begin{aligned}
\int\limits_G F(g)\,dg &= 2^{n(n+3)/2}\,\pi^{n(n+1)/2}\prod_{m=1}^{n}\frac{\Gamma(k - (n+m)/2)}{\Gamma(k - (n-m)/2)} \\
&= 2^{n(n+2)}\,\pi^{n(n+1)/2}\prod_{m=1}^{n}\prod_{j=1}^{m}\frac{1}{2k - n - m - 2 + 2j}.
\end{aligned}
\tag{136}
$$

Next we evaluate the right hand side RHS of (27). With the same calculation as in the proof of Proposition 5.5, we obtain

$$
\mathrm{RHS} = \alpha_n 2^n\int\limits_T\left(\prod_{1\le i<j\le n}(t_i^2 - t_j^2)\right)\left(\prod_{j=1}^{n}t_j\right)^{1-2k}\,d\mathbf{t},
\tag{137}
$$

with T as in (67). Thus, by Lemma 5.4,

$$\text{RHS} = \alpha_n 2^n \prod_{m=1}^{n} (m-1)! \prod_{j=1}^{m} \frac{1}{2k-n-m-2+2j}.$$

(138)

Our assertion now follows by comparing (136) and (138). ∎

A.2   The Iwasawa measure

In this section we will prove the integration formula (85) in the proof of Lemma 5.6. It is well known that the formula holds up to a constant, but we would like to know this constant precisely.

Let T be the group of real upper triangular $n \times n$ matrices with positive diagonal entries. Then $T = AN_1$, where A is the group of $n \times n$ diagonal matrices with positive diagonal entries, and N_1 is the group of $n \times n$ upper triangular matrices with 1's on the diagonal. We will put the following left-invariant Haar measure on T,

$$\int_T \phi(h)\, dh = 2^n \int_A \int_{N_1} \phi(an)\, dn\, da,$$

(139)

where dn is the Lebesgue measure, and $da = \frac{da_1}{a_1} \ldots \frac{da_n}{a_n}$ for $a = \mathrm{diag}(a_1, \ldots, a_n)$. Let P_+ be the set of positive definite $n \times n$ matrices. We endow P_+ with the Lebesgue measure dY, which also occurs in (12).

A.2 Lemma. *The map*

$$\alpha : T \longrightarrow P_+, \qquad h \longmapsto h\,{}^t h$$

(140)

is an isomorphism of smooth manifolds. For a measurable function φ on P_+, we have

$$\int_T \varphi(h\,{}^t h)\, dh = \int_{P_+} \varphi(Y) \det(Y)^{-\frac{n+1}{2}}\, dY.$$

(141)

Here dh is the measure defined by (139), and dY is the Lebesgue measure on the open subset P_+ of $\mathrm{Sym}_n(\mathbb{R})$.

Proof. By Proposition 5.3 on page 272/273 of [22], the map (140) is a diffeomorphism. The proof of formula (141) is an exercise using the tranformation formula from multivariable calculus. ∎

A.3 Proposition. *Let dh be the Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$ characterized by the property (12). Then, for any measurable function F on $\mathrm{Sp}_{2n}(\mathbb{R})$,*

$$\int_{\mathrm{Sp}_{2n}(\mathbb{R})} F(h)\, dh = 2^n \int_A \int_N \int_K F(ank)\, dk\, dn\, da,$$

(142)

where N is the unipotent radical of the Borel subgroup, dn is the Lebesgue measure, $A = \{\mathrm{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) : a_1, \ldots, a_n > 0\}$, and $da = \frac{da_1}{a_1} \ldots \frac{da_n}{a_n}$.

Proof. It is well known that the right hand side defines a Haar measure $d'h$ on $\mathrm{Sp}_{2n}(\mathbb{R})$. To prove that $d'h = dh$, it is enough to consider K-invariant functions F. For such an F, let f be the corresponding function on $\mathbb{H}_n$, i.e., $f(gI) = F(g)$. Let N_1 and N_2 be as in (87), so that $N = N_1 N_2$. By identifying elements of A and N_1 with their upper left block, our notations are consistent with those used in (139). We calculate

$$\int\limits_G F(g)\, d'g = 2^n \int\limits_A \int\limits_{N_1} \int\limits_{N_2} \int\limits_K F(an_1 n_2 k)\, dk\, dn_2\, dn_1\, da$$

$$= 2^n \int\limits_A \int\limits_{N_1} \int\limits_{N_2} F(n_2 a n_1) \det(a)^{-(n+1)}\, dn_2\, dn_1\, da.$$

In the last step we think of a as its upper left $n \times n$ block when we write $\det(a)$. Continuing, we get

$$\int\limits_G F(g)\, d'g = 2^n \int\limits_A \int\limits_{N_1} \int\limits_{\mathrm{Sym}_n(\mathbb{R})} F\left(\left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right] a n_1\right) \det(a)^{-(n+1)}\, dX\, dn_1\, da$$

$$= 2^n \int\limits_A \int\limits_{N_1} \int\limits_{\mathrm{Sym}_n(\mathbb{R})} f(X + a n_1 I) \det(a)^{-(n+1)}\, dX\, dn_1\, da$$

$$= 2^n \int\limits_A \int\limits_{N_1} \int\limits_{\mathrm{Sym}_n(\mathbb{R})} f(X + i(a n_1)\, {}^t(a n_1)) \det(a)^{-(n+1)}\, dX\, dn_1\, da.$$

In the last step, again, we identify a and n_1 with their upper left blocks. By (139), we obtain

$$\int\limits_G F(g)\, d'g = \int\limits_T \int\limits_{\mathrm{Sym}_n(\mathbb{R})} f(X + ih\, {}^t h) \det(h)^{-(n+1)}\, dX\, dh$$

$$= \int\limits_T \int\limits_{\mathrm{Sym}_n(\mathbb{R})} f(X + ih\, {}^t h) \det(h\, {}^t h)^{-\frac{n+1}{2}}\, dX\, dh$$

$$= \int\limits_{P_+} \int\limits_{\mathrm{Sym}_n(\mathbb{R})} f(X + iY) \det(Y)^{-(n+1)}\, dX\, dY.$$

where in the last step we applied Lemma A.2. Using (12), we see $\int\limits_G F(g)\, d'g = \int\limits_G F(g)\, dg$, as asserted. $\blacksquare$

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