International Journal of Number Theory

An explicit lifting construction of CAP forms on $\mathbf{O(1,5)}$

Hiro-aki Narita

Department of mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Ohkubo
Shinjuku, Tokyo 169-8555, Japan,
hnarita@waseda.jp

Ameya Pitale

Department of Mathematics, University of Oklahoma
Norman, Oklahoma, USA
apitale@ou.edu

Siddhesh Wagh

Department of Mathematics, Bar Ilan University
Ramat Gan, Israel
siddesw@biu.ac.il

Received (1 April 2022)
Accepted (6 November 2022)

We explicitly construct non-tempered cusp forms on the orthogonal group O(1,5) of signature $(1+,5-)$. Given a definite quaternion algebra B over $\mathbb{Q}$, the orthogonal group is attached to the indefinite quadratic space of rank 6 with the anisotropic part defined by the reduced norm of B. Our construction can be viewed as a generalization of the previous work by the first two authors joint with Masanori Muto to the case of any definite quaternion algebras, for which we note that the work just mentioned takes up the case where the discriminant of B is two. Unlike the previous work the method of the construction is to consider the theta lifting from Maass cusp forms to $O(1,5)$, following the formulation by Borcherds. The cuspidal representations generated by our cusp forms are studied in detail. We determine all local components of the cuspidal representations and show that our cusp forms are CAP forms.

Keywords: Theta lifting; non-tempered cusp forms; CAP forms.

Mathematics Subject Classification 2010: 11F27, 11F55, 11F70

1. Introduction

Since the discovery of counterexamples to the Ramanujan conjecture by Saito-Kurokawa [20] and Howe-Piatetskii-Shapiro [11] et al. we have known that one has to take into consideration the existence of cuspidal representations with a non-tempered local component towards the classification of cuspidal representations. We call such cusp forms non-tempered. The representation theoretic study of [20] and

[11] by Piatetskii Shapiro [27] leads to the notion of CAP representations, namely cuspidal representations nearly equivalent to irreducible constituents of parabolic inductions (see Definition 8.5). There have been active representation theoretic studies on CAP representation (cf. Soudry [37], Gelbart-Rogawski [7], Rallis-Schiffmann [31], Ginzburg [8], Ginzburg-Rallis-Soudry [10], Ginzburg-Jiang-Soudry [9], et al). The CAP representations are expected to exhaust a large class of non-tempered cusp forms.

We are motivated by a non-holomorphic real analytic construction of non-tempered cusp forms. Our study began with [23], which provided a non-tempered cusp form on $\mathrm{GL}_2(B)$ for a division quaternion algebra B over $\mathbb{Q}$ with discriminant 2. This was inspired by the paper [28] of the second named author, whose tool is the converse theorem by Maass [22]. We have also constructed non-tempered cusp forms on the orthogonal group $\mathrm{O}(1, 8n+1)$ in [21] by Borcherds' theta lifting (cf. [3]). Note that there is an accidental isomorphism relating $\mathrm{PGL}_2(B)$ with $\mathrm{SO}(1,5)$ or $\mathrm{O}(1,5)$ as $\mathbb{Q}$-algebraic groups (cf. Section 2.3), where $\mathrm{SO}(1,5)$ and $\mathrm{O}(1,5)$ are attached to the quadratic form of signature $(1+, 5-)$ whose anisotropic part is defined by the reduced norm of B. Following the approach of [21] this paper constructs non-tempered cusp forms on $\mathrm{O}(1,5)$ for the case of any definite quaternion algebra B, namely with no restriction on the discriminants of B. They turn out to satisfy the CAP properties. The lifting constructions from smaller groups are typical ways to find examples of non-tempered cusp forms. For references in this direction we cite Oda [26], Rallis-Schiffmann [30], Ikeda [12,13], Ikeda-Yamana [14], Yamana [40,41] and Kim-Yamauchi [18] et al.

Let us now describe the main results of the paper. Let d_B be the discriminant of a definite quaternion algebra B over $\mathbb{Q}$. For a maximal order $\mathcal{O}$ of B, let $\mathcal{O}'$ be the dual lattice of $\mathcal{O}$ with respect to the reduced trace of B. We denote by Q_{A_0} (cf. Sections 2) the quadratic form attached to the reduced norm of B. Let Γ be the stabilizer of the lattice $\mathcal{O} \oplus \mathbb{Z}^2$ in the $\mathbb{Q}$-rational points of the orthogonal group defined by $Q_A = Q_{A_0} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (cf. Sections 2). The space of modular forms on the 5-dimensional hyperbolic space with respect to Γ (respectively the space of Maass cusp forms of level d_B) is denoted by $\mathcal{M}(\Gamma, \sqrt{-1}r)$ (respectively $S(\Gamma_0(d_B), r)$) and any $F \in \mathcal{M}(\Gamma, \sqrt{-1}r)$ has the Fourier expansion (see Section 2 for details on notations)

$$F(n(x)a_y) = \sum_{\beta \in \mathcal{O}'} A(\beta) y^2 K_{\sqrt{-1}r}(4\pi \sqrt{Q_{A_0}(\beta)} y) e({}^t\beta A_0 x). \tag{1.1}$$

Our construction is given by a theta lift F_f from Maass cusp forms f of level d_B, with Fourier coefficients $A(\beta)$ explicitly described in terms of Fourier coefficients $c(m)$ of f.

To describe the formula for $A(\beta)$, let us introduce the set of the primitive elements as follows:

$$\mathcal{O}'_{\mathrm{prim}} := \{\beta \in \mathcal{O}' : \frac{1}{n}\beta \notin \mathcal{O}' \text{ for all positive integers } n > 1\}.$$

Write $\beta \in \mathcal{O}'$ as

$$\beta = \prod_{p|d_B} p^{u_p} n\beta_0, \qquad u_p \geq 0, n > 0, \gcd(n, d_B) = 1 \text{ and } \beta_0 \in \mathcal{O}'_{\text{prim}}.$$

Let $q_{\beta_0} = q_{\mu_{\beta_0}}$ be the denominator of the simple fraction for the reduced norm of β_0 (cf. Section 3.3), which is a divisor of d_B. For $p|d_B$, set

$$\delta_p = \begin{cases} 0 & \text{if } p|q_{\beta_0}; \\ 1 & \text{if } p \nmid q_{\beta_0}. \end{cases}$$

Let us assume that the Maass cusp form f has the Atkin-Lehner eigenvalue ϵ_p at $p|d_B$ and has the trivial central character. Define

$$A(\beta) := \sqrt{Q_{A_0}(\beta)} \sum_{p|d_B} \sum_{t_p=0}^{2u_p+\delta_p} \sum_{d|n} c\Big(\frac{-Q_{A_0}(\beta)}{\prod_{p|d_B} p^{t_p-1} d^2}\Big) \prod_{p|d_B} (-\varepsilon_p)^{t_p-1}. \tag{1.2}$$

Putting together the results obtained in Theorem 4.4, Proposition 4.5, Proposition 5.2, Theorem 6.2, Theorem 6.5 and Theorem 7.1 we have the following result.

Theorem 1.1. *Let B be a definite quaternion division algebra with discriminant d_B, which is square-free by definition, and let $\mathcal{O}$ be any maximal order of B. Let $f \in S(\Gamma_0(d_B), r)$, be an Atkin-Lehner eigenfunction with eigenvalues ϵ_p for $p|d_B$. Let F_f be a function on the 5-dimensional hyperbolic space given by the Fourier expansion (1.1) with coefficients $A(\beta)$ given in (1.2). Then the following is true:*

i) F_f is a non-zero, cusp form in $\mathcal{M}(\Gamma, \sqrt{-1}r)$ for all non zero f.
ii) Suppose further that f is a Hecke eigenform with eigenvalues λ_p for all $p \nmid d_B$. Then F_f is also an eigenfunction for the Hecke algebra $\mathcal{H}_p$ for all primes p.
iii) For $p \nmid d_B$, let $\mu_i, i = 1, 2, 3$ be the Hecke eigenvalues for F_f corresponding to the three generators $C_3^{(i)}, i = 1, 2, 3$ of $\mathcal{H}_p$. Then we have

$$\mu_1 = p^2(\lambda_p^2 - 2) + pf_{2,1} = p^2(\lambda_p^2 + p + p^{-1})$$

$$\mu_i = |R_2^{(i-1)}| \left(\mu_1 - \frac{p^{i-1} - 1}{p^i - 1} f_{3,1}\right), (i = 2, 3)$$

See (6.3) and (6.4) for the definition of $f_{2,1}$, $f_{3,1}$ and $|R_2^{(i-1)}|$.
iv) Suppose that f is a newform. For $p|d_B$, let μ be the Hecke eigenvalue of F_f for the Hecke operator $C_1^{(1)}$, which generates $\mathcal{H}_p$. Then we have

$$\mu = p^3 + p^2 - p + 1.$$

By adelizing our explicit lifts in terms of their Fourier expansion we can develop their Hecke theory to obtain the theorem above. This also enables us to understand the cuspidal representations generated by the lifts explicitly.

Theorem 1.2 (Theorem 8.4, Proposition 8.6, Proposition 8.7). *Suppose that the Maass cusp form f is a newform with the trivial central character, and Hecke eigenvalues λ_p for primes $p \nmid d_B$. Let π be the cuspidal representation of $\mathrm{O}(1,5)(\mathbb{A})$ generated by the lift F_f from f.*

(1) The representation π is irreducible and decomposes into the restricted tensor product $\pi = \otimes'_{v\le\infty}\pi_v$ of irreducible admissible representations π_v.

(2) For $v = p < \infty$, if $p \nmid d_B$ then π_p is the spherical constituent of the unramified principal series representation of $\mathrm{O}(1,5)(\mathbb{Q}_p) \simeq \mathrm{O}(3,3)(\mathbb{Q}_p)$ with the Satake parameter

$$\mathrm{diag}\left(\left(\frac{\lambda_p+\sqrt{\lambda_p^2-4}}{2}\right)^2, p, 1, 1, p^{-1}, \left(\frac{\lambda_p+\sqrt{\lambda_p^2-4}}{2}\right)^{-2}\right).$$

(3) For $v = p < \infty$, if $p \mid d_B$ then π_p is the spherical constituent of the spherical representation $I(\chi)$ of $\mathrm{O}(1,5)(\mathbb{Q}_p)$ induced from the unramified character χ of the split torus of $\mathrm{O}(1,5)(\mathbb{Q}_p)$ isomorphic to $\mathbb{Q}_p^\times$ with $\chi(p) = p$.

(4) For every finite prime p, π_p is non-tempered. Suppose that the Selberg conjecture on the minimal Laplace eigenvalue holds for f. Then π_∞ is tempered.

(5) The cuspidal representation is a CAP representation associated with some explicit parabolic induction of $\mathrm{O}(3,3)(\mathbb{A})$.

(6) Let σ denote the cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f. Let $\Pi = \mathrm{Ind}_{P_{2,2}(\mathbb{A})}^{\mathrm{GL}_4(\mathbb{A})}(|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma)$, with the parabolic subgroup $P_{2,2}$ of GL_4 with Levi part $\mathrm{GL}_2\times\mathrm{GL}_2$. By $L(F_f, \mathrm{std}, s)$ (respectively $L(\Pi, \wedge, s)$) we denote the standard L-function for the lift F_f (respectively exterior square L-function of Π). We have

$$L(F_f, \mathrm{std}, s) = L(\Pi, \wedge, s) = L(\mathrm{sym}^2(f), s)\zeta(s-1)\zeta(s)\zeta(s+1),$$

Let us note that a priori, the lift F_f depends on the discriminant d_B of the quaternion algebra B, the Atkin-Lehner eigenform $f \in S(\Gamma_0(d_B), r)$ and the maximal order $\mathcal{O}$ in B. The above theorem shows that the local components of the representation π generated by F_f are in fact independent of the maximal order $\mathcal{O}$ and the Atkin-Lehner eigenvalues of f. It is interesting that the explicit Fourier coefficients $A(\beta)$ clearly depend on the maximal order $\mathcal{O}$ and the Atkin-Lehner eigenvalues ϵ_p for $p|d_B$, while the local components of the cuspidal automorphic representation do not. A multiplicity one theorem for $\mathrm{O}(1,5)$ would imply that different maximal orders would give lifts which are different vectors in the same cuspidal automorphic representations. Such a multiplicity one theorem is not currently available but is expected since we have the multiplicity one theorem by Badulescu and Renard [1] for the group $\mathrm{PGL}_2(B)$.

There are a few significant differences between the results and methods of this paper as compared to our previous work in [21,23]. In [23], we restricted ourselves to the case $d_B = 2$. Here the discrete group Γ was generated by translations and an inversion. The Maass converse theorem [22] gives a criterion for modularity with respect to such groups, and we used it to show that the proposed lift in [23] is a modular form. A situation for which we know that the discrete subgroup Γ has such generators is when $\mathcal{O}$ satisfies the Euclidean property with respect to the reduce norm. Using [25] we can prove that this happens only when the discriminant d_B equals $2,\ 3$ or 5. In this case, we have obtained the proof of modularity of our lift using the Maass converse theorem, but we have not included it in this article since we are interested in general Bs. Instead, to prove modularity, we have used the more general method of Borcherds theta lifts as in [21].

In [21], we were constructing lifts to modular forms on $\mathrm{O}(1, 8n+1)$ starting from Maass forms of full level. For the lifting in Theorem 1.1 above, we need to consider Maass forms with square-free level d_B. For the Borcherds theta lift method to work, an initial step is to transition from scalar valued Maass forms with level d_B to vector valued modular forms with respect to the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$. We work out the corresponding vector valued modular forms and obtain explicit formulas for their Fourier coefficients. This is an active area of research, and the explicit formulas for the Fourier coefficients in the square-free case might be of independent interest.

The explicit formula (1.2) for the Fourier coefficients $A(\beta)$ in Proposition 4.5 needs subtle understanding of the structure of the discriminant form $\mathcal{O}'/\mathcal{O}$ to determine which elements of $\mathcal{O}'$ correspond to which cusps of $\Gamma_0(d_B)$. Furthermore, we remark that, in [21] and [23], we showed the non-vanishing of the lifts by reducing the non-vanishing of $\mathbb{A}(\beta)$ to that of $c(-M)$ for a suitable positive integer M. For the proof we used the explicit formula for $A(\beta)$ together with the surjectivity for the norm map of some special lattice to the set of non-negative integers. However, since the maximal order $\mathcal{O}$ is arbitrary and such surjectivity is not always true for a general $\mathcal{O}$, even the explicit nature of the formula for $A(\beta)$ is not sufficient to obtain non-vanishing. For the non-vanishing of the lift from Theorem 1.1, we could perhaps use Bhargava's 15 Theorem [2] to show that the norm map is surjective for special cases of maximal orders. But for obtaining the theorem in full generality we use another approach using a simple idea from linear algebra. This requires us to first show that the map $f \to F_f$ takes Hecke eigenforms to Hecke eigenforms. We then see the non-vanishing of $A(1)$ of F_f for a non-zero Hecke eigenform f. For the non-vanishing, it turns out to be enough to show that, when f runs over a Hecke eigenbasis of $S(\Gamma_0(d_B), r)$, the lifts F_f distinguish each other by their Hecke eigenvalues at just one prime $p \nmid d_B$, which lead us to prove that the map $f \to F_f$ is a linear injection from $S(\Gamma_0(d_B), r)$. We should remark that there is a well known approach to the non-vanishing of theta lifts using the inner product formula initiated by Rallis [29]. Our method is very different and elementary.

To obtain the Hecke theory, we use the work of Sugano [38]. The case of $p \nmid d_B$ follows directly as in [21]. The Hecke theory for primes $p|d_B$ requires a detailed analysis of the non-split group and makes use of the explicit formula of the Fourier coefficients $A(\beta)$.

Let us explain the outline of the paper. In Section 2 we begin with the review on the orthogonal groups over which we work. This section includes fundamental facts on definite quaternion algebras and accidental isomorphims necessary for the coming discussion. Section 3 is devoted to a detailed study on vector-valued modular forms. This section includes an explicit description of vector-valued forms lifted from Maass cusp forms with square-free levels, which is indispensable for deducing an explicit formula for Fourier coefficients of our lifts. In Section 4 we formulate our lifts as the theta lifts to $O(1,5)$ in the non-adelic setting and provide their explicit formula for the Fourier coefficients. The lifts are proved to be cuspidal in Section 5.

To obtain the representation theoretic aspect of our lifts we adelize them and discuss their Hecke theory in Section 6. In Section 7 we obtain the non-vanishing of our lifts by virtue of the study on the Hecke theory. In Section 8 we have a detailed understanding of the cuspidal representations generated by our lifts, all of whose local components are determined explicitly. As a result our lifts are non-tempered at every non-archimedean place while they are tempered at the archimedean place under the assumption that the Selberg conjecture on the minimal Laplace eigenvalue holds for Maass cusp forms f. The lifts are then proved to be CAP forms attached to some explicitly given parabolic induction for the split orthogonal group $\mathrm{O}(3,3)$. Section 8 ends with an explicit formula for the global standard L-functions of the lifts from Maass cusp forms, whose statement is given as Proposition 8.7. The definition follows Sugano [38, Section 7 (7.6)]. Proposition 8.7 also shows that our global standard L-function coincides with the exterior square L-function for some parabolic induction of GL_4.

2. Preliminaries

In this section, we give the definitions of orthogonal groups, modular forms and quaternion algebras. We also give details on certain accidental isomorphisms.

2.1. *Orthogonal groups and modular forms*

Let $A_0 \in M_4(\mathbb{Q})$ be a positive definite symmetric matrix, and put $A = \begin{bmatrix} & & 1 \\ & -A_0 & \\ 1 & & \end{bmatrix}$.
By $\mathcal{G}$ and $\mathcal{H}$ we denote the $\mathbb{Q}$-algebraic groups defined by

$$\mathcal{G}(\mathbb{Q}) = \{g \in \mathrm{GL}_6(\mathbb{Q}) \mid {}^t gAg = A\}, \quad \mathcal{H}(\mathbb{Q}) = \{h \in \mathrm{GL}_4(\mathbb{Q}) \mid {}^t hA_0h = A_0\}$$

respectively. Both $\mathcal{G}$ and $\mathcal{H}$ are referred to as orthogonal groups. We introduce the standard proper $\mathbb{Q}$ parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ defined by the Levi decomposition

$\mathcal{P} = \mathcal{N}\mathcal{L}$ with

$$\mathcal{N}(\mathbb{Q}) = \left\{ n(x) = \begin{pmatrix} 1 & {}^t x A_0 & \frac{1}{2}{}^t x A_0 x \\ & 1_4 & x \\ & & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{Q}^4 \right\},$$

$$\mathcal{L}(\mathbb{Q}) = \left\{ a_\alpha = \begin{pmatrix} \alpha & & \\ & h & \\ & & \alpha^{-1} \end{pmatrix} \;\middle|\; \alpha \in \mathbb{Q}^\times,\ h \in \mathcal{H}(\mathbb{Q}) \right\}.$$

Assume that L_0 is a maximal even integral lattice in $\mathbb{Q}^4$ with respect to A_0. We put

$$L := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x, z \in \mathbb{Z},\ y \in L_0 \right\} = L_0 \oplus \mathbb{Z}^2.$$

This is a maximal lattice with respect to A. We let $\Gamma := \{\gamma \in \mathcal{G}(\mathbb{Q}) \mid \gamma L = L\}$.

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_f$ be the set of finite adeles in $\mathbb{A}$. We consider the adelizations of the $\mathbb{Q}$-algebraic groups above, denoted by $\mathcal{G}(\mathbb{A})$, $\mathcal{H}(\mathbb{A})$, $\mathcal{P}(\mathbb{A})$, $\mathcal{N}(\mathbb{A})$ and so on. Let $L_p := L \otimes \mathbb{Z}_p$ and $L_{0,p} := L_0 \otimes \mathbb{Z}_p$ and we put $K_f := \prod_{p<\infty} K_p$ and $U_f := \prod_{p<\infty} U_p$ with

$$K_p := \{k \in \mathcal{G}(\mathbb{Q}_p) \mid kL_p = L_p\}, \quad U_p := \{u \in \mathcal{H}(\mathbb{Q}_p) \mid uL_{0,p} = L_{0,p}\}$$

for each finite prime $p < \infty$. Let K_∞ be the maximal compact subgroup of $\mathcal{G}(\mathbb{R})$ given by

$$\left\{ g \in \mathcal{G}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} 1 & & \\ & A_0 & \\ & & 1 \end{pmatrix} g = \begin{pmatrix} 1 & & \\ & A_0 & \\ & & 1 \end{pmatrix} \right\}.$$

With $A_\infty := \left\{ a_y = \begin{pmatrix} y & & \\ & 1_4 & \\ & & y^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}^+ \right\}$ the Iwasawa decomposition $\mathcal{G}(\mathbb{R}) = \mathcal{N}(\mathbb{R}) A_\infty K_\infty$ gives us the 5-dimensional hyperbolic space $\mathbb{H}_5$ as follows.

$$\mathbb{R}^4 \times \mathbb{R}^+ \ni (x, y) \mapsto n(x) a_y \in \mathcal{G}(\mathbb{R})/K_\infty.$$

Definition 2.1. For $r \in \mathbb{C}$ we denote by $\mathcal{M}(\Gamma, r)$ the space of smooth functions F on $\mathcal{G}(\mathbb{R})$ satisfying the following conditions:

i) $\Omega \cdot F = \frac{1}{8}\left(r^2 - 4\right) F$, where Ω is the Casimir operator defined in [21, (2.3)],
ii) for any $(\gamma, g, k) \in \Gamma \times \mathcal{G}(\mathbb{R}) \times K_\infty$, we have $F(\gamma g k) = F(g)$,
iii) F is of moderate growth.

As usual we say that $F \in \mathcal{M}(\Gamma, r)$ is a cusp form if it vanishes at all the cusps of Γ.

From Proposition 2.3 of [21], we see that a cusp form F in $\mathcal{M}(\Gamma, r)$ has the Fourier expansion

$$F(n(x)a_y) = \sum_{\beta \in L_0' \setminus \{0\}} A(\beta) y^2 K_r(4\pi \sqrt{Q_{A_0}(\beta)} y) e({}^t\beta A_0 x), \tag{2.1}$$

with the dual lattice L_0' of L_0. Here, Q_{A_0} is the quadratic form corresponding to A_0.

2.2. *Quaternion algebras*

We want to restrict to the case where the lattice L_0 from the previous section corresponds to maximal orders in division quaternion algebras. In this section, we will provide the relevant information about quaternion algebras, maximal orders and their duals. A good reference is the book [39] by Jon Voight. Let B be a definite division quaternion algebra over $\mathbb{Q}$, given by $\mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$, with $i^2 = a, j^2 = b, ij = -ji = k$. Let us denote the standard involution on B by $\alpha \mapsto \bar{\alpha}$. Let the trace and norm be defined by $\mathrm{tr}(\alpha) = \alpha + \bar{\alpha}$ and $\mathrm{Nrd}(\alpha) = \alpha\bar{\alpha}$. Assume that B has discriminant $d_B = N$. Hence, N is a square-free integer with an odd number of prime factors.

Let $\mathcal{O}$ be any maximal order in B. Let A_0 be the gram matrix of $\mathcal{O}$ with respect to some basis, so that $\mathcal{O} \simeq (\mathbb{Z}^4, A_0)$. Let Q_{A_0} be the quadratic form given by $Q_{A_0}(x) = \frac{1}{2} {}^t x A_0 x$ for $x \in \mathbb{Z}^4$, and B_{A_0} be the corresponding bilinear form. Note that if $\alpha, \beta \in \mathcal{O}$ get mapped to $x, y \in \mathbb{Z}^4$, then $\mathrm{Nrd}(\alpha) = Q_{A_0}(x)$ and $\mathrm{tr}(\alpha\bar{\beta}) = B_{A_0}(x, y)$. Let

$$L = \{ \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} : a, b \in \mathbb{Z}, \alpha \in \mathcal{O} \}.$$

Then $L \simeq (\mathbb{Z}^6, A)$, with $A = \begin{bmatrix} & & 1 \\ & -A_0 & \\ 1 & & \end{bmatrix}$. Then $Q_A(a, x, b) = ab - Q_{A_0}(x)$. Hence, the signature of L is $(1, 5)$. The bilinear form B_A on L is given by

$$B_A(x, x) = 2Q_A(x),$$

and

$$B_A(x, y) = \frac{1}{2}(B_A(x + y, x + y) - B_A(x, x) - B_A(y, y)) = {}^t x A y$$

for $x, y \in L$. We will be considering the orthogonal groups $\mathcal{G}$ and $\mathcal{H}$ with respect to the above matrices A and A_0.

Define the dual of $\mathcal{O}$ by

$$\mathcal{O}' \coloneqq \{\alpha \in B(\mathbb{Q}) : \mathrm{tr}(\alpha\mathcal{O}) \subset \mathbb{Z}\}.$$

Let us collect some facts about $\mathcal{O}$ and $\mathcal{O}'$.

i) Since $\mathcal{O}$ is maximal, we can see that $\mathcal{O} = \{\alpha \in \mathcal{O}' : \mathrm{Nrd}(\alpha) \in \mathbb{Z}\}$.

ii) Let the discriminant $\mathrm{disc}(\mathcal{O})$ be as in [39, (15.1.2)]. We have $\mathrm{disc}(\mathcal{O}) = N^2$, since $\mathcal{O}$ is a maximal order [39, Theorem 15.5.5]. We also have [39, Lemma 15.6.17]

$$\mathrm{disc}(\mathcal{O}) = [\mathcal{O}' : \mathcal{O}] = N^2.$$

iii) Define

$$(\mathcal{O}')^{-1} := \{\alpha \in B(\mathbb{Q}) : \mathcal{O}'\alpha\mathcal{O}' \subset \mathcal{O}'\}.$$

By [39, Proposition 16.5.8], we have $(\mathcal{O}')^{-1}\mathcal{O}' = \mathcal{O}$. Further, we also have [39, Equation 16.8.4]

$$\mathrm{Nrd}((\mathcal{O}')^{-1}) = \text{ ideal generated by } \mathrm{Nrd}(\alpha) \text{ for all } \alpha \in (\mathcal{O}')^{-1} = N\mathbb{Z}.$$

This gives us

$$\mathrm{Nrd}(\mathcal{O}') = \frac{1}{N}\mathbb{Z}.$$

iv) For a prime number p, let $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$ and $\mathcal{O}'_p = \mathcal{O}' \otimes \mathbb{Z}_p$. It is known that $\mathcal{O}_p$ is a maximal order in $B_p = B \otimes \mathbb{Q}_p$. For $p \nmid N$, B_p is isomorphic to $M_2(\mathbb{Q}_p)$. Up to conjugation, there is a unique maximal order in B_p given by $M_2(\mathbb{Z}_p)$, which is its own dual.

v) For $p|N$, B_p is a division algebra. From [36, Theorem 5.13], we have the following information on the local maximal order and its dual.

- We have a unique maximal order $\mathcal{O}_p$ in B_p given by $\{\alpha \in B_p : \mathrm{Nrd}(\alpha) \in \mathbb{Z}_p\}$.
- Let $\mathfrak{P} := \{\alpha \in B_p : \mathrm{Nrd}(\alpha) \in p\mathbb{Z}_p\}$. Then we have

$$\mathfrak{P}^m = \{\alpha \in B_p : \mathrm{Nrd}(\alpha) \in p^m\mathbb{Z}_p\}$$

  for $m \in \mathbb{Z}$, $\quad p\mathcal{O}_p = \mathfrak{P}^2$, and $\mathcal{O}'_p = \mathfrak{P}^{-1}$.
- Let $K_p \subset B_p$ be the unique non-trivial unramified extension of $\mathbb{Q}_p$. We have

$$K_p = \begin{cases} \mathbb{Q}_2(\sqrt{5}) & \text{if } p = 2; \\ \mathbb{Q}_p(\sqrt{-1}) & \text{if } p \equiv 3, 7 \pmod 8; \\ \mathbb{Q}_p(\sqrt{2}) & \text{if } p \equiv 5 \pmod 8; \\ \mathbb{Q}_p(\sqrt{q}) & \text{if } p \equiv 1 \pmod 8,\ q \equiv 3 \pmod 4,\ \left(\frac{p}{q}\right) = -1. \end{cases}$$

  Let $\mathcal{O}_{K_p}$ be the ring of integers of K_p. Then there exists $w_p \in B_p$ such that $w_p^2 = p$ and $B_p = K_p + w_pK_p$, $\mathcal{O}_p = \mathcal{O}_{K_p} + w_p\mathcal{O}_{K_p}$ and $\mathfrak{P} = w_p\mathcal{O}_p$. Hence, $\mathcal{O}'_p = w_p^{-1}\mathcal{O}_p = \mathcal{O}_{K_p} + w_p^{-1}\mathcal{O}_{K_p}$.
- We have

$$\mathcal{O}'_p/\mathcal{O}_p \simeq w_p^{-1}\mathcal{O}_{K_p}/\mathcal{O}_{K_p} \simeq \langle w_p^{-1}\rangle \times \langle uw_p^{-1}\rangle \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z},$$

where

$$u = \begin{cases} \sqrt{5} & \text{if } p = 2; \\ \sqrt{-1} & \text{if } p \equiv 3 \pmod 4; \\ \sqrt{2} & \text{if } p \equiv 5 \pmod 8; \\ \sqrt{q} & \text{if } p \equiv 1 \pmod 8. \end{cases}$$

2.3. *Accidental isomorphisms*

For a quaternion algebra E over $\mathbb{Q}$ with the reduced norm N_E we view (E, N_E) as a rank 4 quadratic space over $\mathbb{Q}$. This gives rise to the rank 6 quadratic space $V_E := (E, N_E) \oplus \mathbb{H}$ with the hyperbolic space $\mathbb{H}$. For the subsequent argument we will need the two well-known accidental isomorphisms

$$E^\times \times E^\times / \{(z, z) \mid z \in \mathrm{GL}_1\} \simeq \mathrm{GSO}(E, N_E),$$
$$\mathrm{GL}_2(E) \times \mathrm{GL}_1 / \{(z \cdot 1_4, z^{-2}) \mid z \in \mathrm{GL}_1\} \simeq \mathrm{GSO}(V_E)$$

as $\mathbb{Q}$-algebraic groups (cf. [5, Section 3]).

Let $E := M_2$ be the matrix algebra of degree two over $\mathbb{Q}$. The group on the right hand side of the first isomorphism is the similitude group defined by the determinant form of M_2. We denote this by $\mathrm{GSO}(2,2)$ in view of the signature of the quadratic space at the archimedean place. The isomorphism is induced by

$$\mathrm{GL}_2 \times \mathrm{GL}_2 \ni (h_1, h_2) \mapsto M_2 \ni X \mapsto h_1 X h_2^{-1} \in M_2.$$

Let ι be the main involution of M_2. This induces the outer automorphism

$$\mathrm{GL}_2 \times \mathrm{GL}_2 \ni (h_1, h_2) \mapsto (\iota(h_1)^{-1}, \iota(h_2)^{-1}) \in \mathrm{GL}_2 \times \mathrm{GL}_2.$$

We denote this by t. With this t we have an isomorphism

$$\mathrm{GO}(2,2) \simeq \mathrm{GSO}(2,2) \rtimes \langle t \rangle.$$

Regarding the second isomorphism the similitude group on the right hand side is defined by the quadratic form $ab - N_E(X)$ defined on the $\mathbb{Q}$-vector space

$$V_E := \left\{ \begin{pmatrix} a & x \\ \iota(x) & b \end{pmatrix} \mid a,\ b \in \mathbb{Q},\ x \in M_2 \right\}.$$

Since the signature of this quadratic space is $(3+, 3-)$ this group can be denoted by $\mathrm{GSO}(3,3)$. The isomorphism is given by

$$\mathrm{GL}_4 \times \mathrm{GL}_1 \ni (g, z) \mapsto V_E \ni X \mapsto z \cdot g X {}^t\iota(g) \in V_E,$$

where we put $\iota(g) := \begin{pmatrix} \iota(x) & \iota(y) \\ \iota(z) & \iota(w) \end{pmatrix}$ for $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $x, y, z, w \in M_2$.

Of course, we are interested in the case of $E = B$. For this case the similitude groups can be denoted by $\mathrm{GSO}(4)$ and $\mathrm{GSO}(1,5)$ for the first and second isomorphisms respectively.

3. Vector valued modular form

In this section, we will start with a weight 0 Maass form for $\Gamma_0(N)$ and construct a weight $(0,0)$ vector-valued modular form for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on a group algebra of a discriminant form. The main reference for this section is [34].

3.1. *The discriminant form*

As in the previous section, let B be a definite quaternion algebra over $\mathbb{Q}$ with discriminant $d_B = N$, a square-free integer. Let $\mathcal{O}$ be any maximal order of B with $\mathcal{O} \simeq (\mathbb{Z}^4, A_0)$. Let Q_{A_0}, L and A be as in Section 2.2.

Let $\mathcal{O}'$ and L' be the dual of $\mathcal{O}$ and L respectively with respect to bilinear forms B_{A_0} and B_A. We have described the dual $\mathcal{O}'$ in the previous section. We have

$$L' = \{ \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} : a, b \in \mathbb{Z}, \alpha \in \mathcal{O}' \}.$$

Define the discriminant form D by $D = L'/L$. From the description of L' above, we have $D = L'/L = \mathcal{O}'/\mathcal{O}$. This D inherits the quadratic form Q_D and bilinear form B_D (with values in $\mathbb{Q}/\mathbb{Z}$) from those of $\mathcal{O}'$ considered modulo 1. The level of D is the smallest positive integer n such that $nQ_D(\mu) \equiv 0 \pmod 1$ for all $\mu \in D$. Since $\mathrm{Nrd}(\mathcal{O}') = \frac{1}{N}\mathbb{Z}$, we see that the level of D is N. Every discriminant form is an orthogonal direct sum of basic discriminant forms, which are described in Section 3 of [34]. The basic discriminant forms all correspond to the prime divisors of N. Let us write $D = \oplus_{p|N} D_p$, where by Section 2.2, we have

$$D_p = \langle w_p^{-1} \rangle \times \langle u w_p^{-1} \rangle.$$

We have $Q_D(w_p^{-1}) = -1/p$ and $Q_D(uw_p^{-1}) = u^2/p$. When $p = 2$, we see that $Q_D(w_p^{-1}) = Q_D(uw_p^{-1}) = B_D(w_p^{-1}, uw_p^{-1}) = 1/2$. Hence, in the notation of Section 3 of [34], we have $D_2 = 2_{\mathrm{II}}^{-2}$.

Next, suppose p is an odd prime. Since $Q_D(w_p^{-1}) = -1/p$, the basic discriminant form corresponding to $\langle w_p^{-1} \rangle$ is p^{ϵ}, where $\epsilon = \left(\frac{-2}{p}\right)$. On the other hand

$$Q_D(uw_p^{-1}) = \begin{cases} -1/p & \text{if } p \equiv 3 \pmod 4; \\ 2/p & \text{if } p \equiv 5 \pmod 8; \\ q/p & \text{if } p \equiv 1 \pmod 8. \end{cases}$$

If $Q_D(uw_p^{-1}) = a/p$, then $\langle uw_p^{-1} \rangle$ corresponds to the discriminant form $p^{\epsilon'}$, where $\epsilon' = \left(\frac{2a}{p}\right)$. Hence, by Section 3 of [34], we have

$$D_p = \begin{cases} p^{+1} \times p^{+1} = p^{+2} & \text{if } p \equiv 3 \pmod 8; \\ p^{-1} \times p^{-1} = p^{+2} & \text{if } p \equiv 7 \pmod 8; \\ p^{+1} \times p^{-1} = p^{-2} & \text{if } p \equiv 1, 5 \pmod 8. \end{cases} \tag{3.1}$$

We have the following relevant information about D.

i) The level of D is N and $|D| = N^2$.
ii) The signature of D is $\mathrm{sgn}(D) = 1 - 5 \pmod 8 = 4$.
iii) $D = \oplus_{p|N} D_p$, where $D_p = \{\mu \in D : p\mu = 0\}$.
iv) The oddity of D is 4 if N is even, and is 0 if N is odd.

3.2. *Weil representation*

The group algebra $\mathbb{C}[D]$ is a $\mathbb{C}$-vector space generated by the formal basis vectors $\{e_\mu : \mu \in D\}$ with product defined by $e_\mu e_{\mu'} = e_{\mu+\mu'}$. The inner product on $\mathbb{C}[D]$ (anti-linear in the second argument) is defined by $\langle e_\mu, e_{\mu'} \rangle = \delta_{\mu,\mu'}$. Hereafter we will often use the notation

$$e(x) \coloneqq \exp(2\pi\sqrt{-1}x)$$

for $x \in \mathbb{R}$. We will now define a representation ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ by specifying it on the generators of $\mathrm{SL}_2(\mathbb{Z})$ given by $T = \left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$.

$$\rho_D(T)e_\mu = e(Q_D(\mu))e_\mu,$$
$$\rho_D(S)e_\mu = \frac{e(-\mathrm{sgn}(D)/8)}{\sqrt{|D|}} \sum_{\mu' \in D} e(-B_D(\mu,\mu'))e_{\mu'} = -\frac{1}{N} \sum_{\mu' \in D} e(-B_D(\mu,\mu'))e_{\mu'}.$$

This action extends to a unitary representation ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ called the Weil representation of D. The restriction of ρ_D to the congruence subgroup $\Gamma_0(N)$ is given in the next lemma.

Lemma 3.1. *Let* $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ *and* $\mu \in D$. *Then*

$$\rho_D(M)e_\mu = e(bdQ_D(\mu))e_{d\mu}.$$

In particular, we have $\rho_D(M)e_0 = e_0$ *for all* $M \in \Gamma_0(N)$.

Proof. From equation (4.1) of [34] we get, for $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ and $\mu \in D$,

$$\rho_D(M)e_\mu = \chi_D(a)e(bdQ_D(\mu))e_{d\mu}, \text{ where } \chi_D(a) = \left(\frac{a}{|D|}\right)e((a-1)\cdot \mathrm{oddity}(D)/8).$$

Note that $|D| = N^2$ and $\mathrm{oddity}(D) = 4$ if N is even and 0 if N is odd. Hence, for all D, we have that χ_D is the trivial character. This gives the lemma. □

3.3. *Scalar to vector valued modular form*

To construct a vector valued modular form for $\mathrm{SL}_2(\mathbb{Z})$ with values in $\mathbb{C}[D]$, one has to start with a scalar valued modular form of level divisible by the level of D and nebentypus character χ_D. In our case, the level of D is N and the character χ_D is trivial. Hence, we let $S(\Gamma_0(N), r)$ be the space of Maass cusp form of weight 0 with respect to $\Gamma_0(N)$ with Laplace eigenvalue $(r^2+1)/4$. According to the Selberg conjecture on the minimal Laplace eigenvalue for Maass cusp forms, r should be real (cf. [15, Section 11.3 Conjecture]). The Fourier expansion of $f \in S(\Gamma_0(N), r)$ is given by

$$f(u+iv) = \sum_{n\neq 0} c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|v)e(nu).$$

for $\mathfrak{h} := \{u + iv \in \mathbb{C} : v > 0\}$. Define $\mathcal{L}_D(f) : \mathfrak{h} \to \mathbb{C}[D]$ by

$$\mathcal{L}_D(f) = \sum_{M\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f|M\rho_D(M)^{-1}e_0, \tag{3.2}$$

where $(f|M)(\tau) = f(M \cdot \tau) := f((a\tau + b)/(c\tau + d))$ for $M = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$.

Proposition 3.2. *Let $f \in S(\Gamma_0(N), r)$. The function $\mathcal{L}_D(f)$ is well-defined and satisfies*

$$\mathcal{L}_D(f)|\gamma = \rho_D(\gamma)\mathcal{L}_D(f),$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. The well-definedness of $\mathcal{L}_D(f)$ follows from the $\Gamma_0(N)$-invariance of f and Lemma 3.1. The automorphy condition follows from a simple change of variable. □

Let us remark here that if H is an isotropic subgroup of D, then the e_0 term in the definition of $\mathcal{L}_D(f)$ can be replaced by a sum over H. In our case, the only isotropic subgroup of D is the trivial one.

In the remainder of the section, we will obtain a formula for the Fourier expansion of $\mathcal{L}_D(f)$. From page 660 of [33], we have

$$\mathcal{L}_D(f)(\tau) = \sum_{c|N}\sum_{\mu\in D_{\frac{N}{c}}} \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}}\frac{N}{c} g_{\frac{N}{c},j_{\mu,\frac{N}{c}}}(\tau)e_\mu. \tag{3.3}$$

Let us explain the terms appearing in the formula above.

i) For any integer t, set $D_t := \{\mu \in D : t\mu = 0\}$. In our case, for every $t|N$, we have $D_t = \oplus_{p|t} D_p$. Hence, $|D_t| = t^2$ for $t|N$.

ii) We have

$$\xi_c := \left(\frac{-c}{|D_{\frac{N}{c}}|}\right)\prod_{p|\frac{N}{c}} \gamma_p(D),$$

with

$$\gamma_p(p^{\pm 2}) = e(-p\text{-excess}(p^{\pm 2})/8) \text{ if } p \text{ is odd},$$
$$\gamma_2(2_{II}^{\pm 2}) = e(\text{oddity}(2_{II}^{\pm 2})/8).$$

We have p-excess$(p^{\pm 2}) = 2(p-1)+k \pmod 8$ where $k = 4$ if the sign is $-$ and $k = 0$ if the sign is $+$. By (3.1), we have $\gamma_p(D_p) = -1$ for all primes p. Hence

$$\xi_c = \prod_{p|\frac{N}{c}}(-1).$$

iii) Finally, let us describe the functions $g_{\frac{N}{c},j}$. For every $c|N$, choose $M_c = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$ such that $d \equiv 1 \pmod{c}$ and $d \equiv 0 \pmod{N/c}$. As in page 658 of [33], we have, for $0 \le j \le N/c$,

$$g_{\frac{N}{c},j}(\tau) = \frac{1}{N/c} \sum_{k \bmod \frac{N}{c}} e\big(\frac{-jk}{N/c}\big)\big(f|M_c T^k\big)(\tau).$$

The integer $j_{\mu,N/c}$ is defined by $(j_{\mu,N/c})/(N/c) \equiv -Q_D(\mu) \pmod{1}$.

Putting all this together, we see that (3.3) now gives us

$$\mathcal{L}_D(f)(\tau) = \sum_{c|N} \prod_{p|\frac{N}{c}} (-1) \frac{1}{N/c} \sum_{k \bmod \frac{N}{c}} \big(f|M_c T^k\big)(\tau) \sum_{\mu \in D_{\frac{N}{c}}} e(kQ_D(\mu))e_\mu. \tag{3.4}$$

To simplify this further, we will assume that f is an eigenfunction of all the Atkin-Lehner operators. For every $c|N$, the Atkin-Lehner operator corresponds to the action on f by the matrix $W_{\frac{N}{c}} \in M_2(\mathbb{Z})$ given by

$$W_{\frac{N}{c}} = \left[\begin{smallmatrix} \frac{N}{c}x & y \\ Nw & \frac{N}{c}x \end{smallmatrix}\right] \text{ with } \det(W_{\frac{N}{c}}) = \frac{N}{c}.$$

Note that $W^2_{\frac{N}{c}} \in Z(\mathbb{Q})\Gamma_0(N)$ with $Z(\mathbb{Q}) := \{z \cdot 1_2 \mid z \in \mathbb{Q}^\times\}$. Now set $\widehat{W}_c := W_{\frac{N}{c}} \left[\begin{smallmatrix} \frac{c}{N} & \\ & 1 \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$.

Since N is square-free, the cusps of $\Gamma_0(N)$ are given by $1/c$ where c runs over all divisors of N. The cusp $1/N$ corresponds to infinity. Given a matrix $M = \left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, it is well known that $M\langle\infty\rangle$ contains the representative $1/c$, where $c = \gcd(c', N)$. Hence, we have $M_c\langle\infty\rangle = \widehat{W}_c\langle\infty\rangle$, which implies that there is a $\gamma_c \in \Gamma_0(N)$ such that $M_c = \gamma_c \widehat{W}_c$.

Proposition 3.3. *Let $f \in S(\Gamma_0(N), r)$ be a Maass cusp form of weight 0 with respect to $\Gamma_0(N)$ with Laplace eigenvalue $(r^2+1)/4$. Assume that f is an eigenfunction of the Atkin-Lehner operators and let $f|W_{\frac{N}{c}} = \varepsilon_{\frac{N}{c}} f$. Then, we have*

$$\mathcal{L}_D(f)(\tau) = \sum_{c|N} \varepsilon_{\frac{N}{c}} \prod_{p|\frac{N}{c}} (-1) \sum_{a \bmod \frac{N}{c}} \sum_{\substack{n \neq 0 \\ n+a \equiv 0 \bmod \frac{N}{c}}} \Big(c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|v\frac{c}{N}) e(nu\frac{c}{N}) \times \sum_{\substack{\mu \in D_{\frac{N}{c}} \\ Q_D(\mu) \equiv \frac{ac}{N} \bmod 1}} e_\mu.\Big)$$

Proof. Since $M_c = \gamma_c \widehat{W}_c$, with $\gamma_c \in \Gamma_0(N)$, we have

$$\begin{aligned}\big(f|M_cT^k\big)(\tau) &= \big(f|\widehat{W}_c T^k\big)(\tau) = \big(f|W_{\frac{N}{c}}\left[\begin{smallmatrix} \frac{c}{N} & \\ & 1 \end{smallmatrix}\right] T^k\big)(\tau) \\ &= \varepsilon_{\frac{N}{c}} \big(f|\left[\begin{smallmatrix} \frac{c}{N} & \frac{kc}{N} \\ & 1 \end{smallmatrix}\right]\big)(\tau) = \varepsilon_{\frac{N}{c}} f(\frac{\tau c}{N} + \frac{kc}{N}) \\ &= \varepsilon_{\frac{N}{c}} \sum_{n \neq 0} c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|v\frac{c}{N}) e(nu\frac{c}{N}) e(nk\frac{c}{N}).\end{aligned}$$

Note that we have

$$\sum_{k \bmod \frac{N}{c}} \sum_{\mu \in D_{\frac{N}{c}}} e(nk\frac{c}{N})e(kQ_D(\mu))e_\mu = \sum_{a \bmod \frac{N}{c}} \sum_{\substack{\mu \in D_{\frac{N}{c}} \\ Q_D(\mu)=ac/N}} \sum_{k \bmod \frac{N}{c}} e(\frac{kc}{N}(n+a))e_\mu.$$

Here, we have used that $\frac{N}{c}Q_D(D_{\frac{N}{c}}) \subset \mathbb{Z}$. We have

$$\sum_{k \bmod \frac{N}{c}} e(\frac{kc}{N}(n+a)) = \begin{cases} \frac{N}{c} & \text{if } n+a \equiv 0 \pmod{\frac{N}{c}}; \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these in (3.4) gives us the formula in the statement of the proposition. □

We want to rewrite the formula for $\mathcal{L}_D(f)(\tau)$ in Proposition 3.3 in the form $\sum_{\mu \in D} f_\mu(\tau)e_\mu$. For this, let us first associate to every $\mu \in D$ an integer $q_\mu | N$ as follows. Since $NQ_D(\mu) \in \mathbb{Z}$, write $Q_D(\mu) = b/N = a/q_\mu$, where $\gcd(a, q_\mu) = 1$. Observe that $\mu \in D_{\frac{N}{c}}$ for every c satisfying $q_\mu | \frac{N}{c} | N$. Hence, we have

$$\mathcal{L}_D(f)(\tau) = \sum_{\mu \in D} \Big(\sum_{c | \frac{N}{q_\mu}} \varepsilon_{\frac{N}{c}} \prod_{p | \frac{N}{c}} (-1) \times \sum_{\substack{n \neq 0 \\ \frac{nc}{N} \equiv -Q_D(\mu) \bmod 1}} c(n) W_{0, \frac{\sqrt{-1}r}{2}}(4\pi |n| v \frac{c}{N}) e(nu\frac{c}{N}) \Big) e_\mu. \tag{3.5}$$

Observe that the coefficient $f_\mu(\tau)$ of e_μ above depends only on $Q_D(\mu)$. Hence, for any $c \in \mathcal{G}(\mathbb{Q})$, we have

$$\mathcal{L}_{cD}(f) = \sum_{\mu \in cD} f'_\mu(\tau)e_\mu \text{ with } f'_\mu = f_{c^{-1}\mu}. \tag{3.6}$$

4. Theta lifts

In this section, we will construct the theta lift of $f \in S(\Gamma_0(N), r)$, N square-free, to an automorphic form on 5-dimensional hyperbolic space as in [3]. Also see [21].

4.1. *Real hyperbolic space as a Grassmanian manifold*

We will follow the construction of the theta lift in Section 3 of [21]. We recall from Section 2 that if $g \in \mathcal{G}(\mathbb{R})$, then we can write

$$g = n(x)a_y k, \text{ where } n(x) = \begin{bmatrix} 1 & {}^t x A_0 & \frac{1}{2} {}^t x A_0 x \\ & 1_4 & x \\ & & 1 \end{bmatrix},\ x \in \mathbb{R}^4,\ a_y = \begin{bmatrix} y & & \\ & 1_4 & \\ & & y^{-1} \end{bmatrix},$$

for $y \in \mathbb{R}^+$, $k \in K_\infty$ where K_∞ is the maximal compact subgroup of $\mathcal{G}(\mathbb{R})$ (cf. Section 2.1) and that

$$\mathbb{R}^4 \times \mathbb{R}^+ \ni (x, y) \mapsto n(x)a_y \in \mathcal{G}(\mathbb{R})/K_\infty$$

gives the 5-dimensional hyperbolic space $\mathbb{H}_5$. Let $V_5 := (\mathbb{R}^6, Q_A)$ and let $\mathcal{D}$ be the Grassmanian of positive oriented lines in the quadratic space V_5. Note that $V_5 = L \otimes \mathbb{R}$, where L was the lattice defined in Section 2.2. We will identify $\mathbb{H}_5$ with a connected component of $\mathcal{D}$ as follows.

$$\mathbb{H}_5 \ni (x, y) \mapsto \nu(x, y) := \frac{1}{\sqrt{2}}{}^t(y + y^{-1}Q_{A_0}(x), -y^{-1}x, y^{-1}) \in V_5$$

satisfying $B_A(\nu(x, y), \nu(x, y)) = 1$. It generates the positive, oriented line $\mathbb{R} \cdot \nu(x, y)$, which is an element in $\mathcal{D}$. In fact, we see that $\mathcal{D}^+ := \{\mathbb{R} \cdot \nu(x, y) \mid (x, y) \in \mathbb{H}_5\}$ is one of the two connected components of $\mathcal{D}$. We now note that the quadratic space V_5 is isometric to $\mathbb{R}^{1,5}$, where $\mathbb{R}^{1,5}$ denotes the real vector space $\mathbb{R}^6$ with the quadratic form

$$Q_{1,5}(x_1, x_2, \cdots, x_6) := \frac{1}{2}\left(x_1^2 - \sum_{j=2}^{6} x_j^2\right).$$

We slightly abuse the notation by using ν to represent the line generated by $\nu(x, y)$. Every line $\nu \in \mathcal{D}^+$ induces an isometry

$$\begin{aligned} \iota_\nu : V_5 &\to \mathbb{R} \cdot \nu \oplus (\nu^\perp, Q_{A_0}|_{\nu^\perp}) \simeq \mathbb{R}^{1,5} \\ \lambda &\mapsto (\iota_\nu^+(\lambda), \iota_\nu^-(\lambda)), \end{aligned}$$

where

$$\iota_\nu^+(\lambda) := B_A(\lambda, \nu)\nu, \;\; \iota_\nu^-(\lambda) := \lambda - \iota_\nu^+(\lambda) \in \nu^\perp$$

are the components of λ. Let us remark here that, if we fix $(x, y) \in \mathbb{H}_5$, then we get a corresponding isometry of V_5 into $\mathbb{R}^{1,5}$ where the one dimensional positive definite subspace is the line generated by $\nu(x, y)$.

Note that $\iota_{\gamma\cdot\nu}^+(\gamma \cdot \lambda) = \gamma \cdot \iota_\nu^+(\lambda)$ for any $\gamma \in \mathcal{G}(\mathbb{R})$ and $\lambda \in V_5$. Next, we collect some facts about the distinguished elements

$$\begin{aligned} &z := {}^t(1, 0_4, 0), \;\; z' := {}^t(0, 0_4, 1), \\ &\mu_0 := -z' + \frac{z_{\nu^+}}{2B_A(z_{\nu^+}, z_{\nu^+})} + \frac{z_{\nu^-}}{2B_A(z_{\nu^-}, z_{\nu^-})} \end{aligned}$$

and their properties. These will be useful later in the Fourier expansion of the theta lift.

Lemma 4.1.

i) We have $B_A(z, z) = B_A(z', z') = 0$ *and* $B_A(z, z') = 1$.

ii) Let $z = (z_{\nu^+}, z_{\nu^-})$ *where* $z_{\nu^+} = \iota_\nu^+(z)$ *and* $z_{\nu^-} = \iota_\nu^-(z)$*. Then*

$$\begin{aligned} z_{\nu^+} = B_A(z, \nu)\nu = \frac{1}{\sqrt{2}y}\nu, \quad z_{\nu^-} = z - z_{\nu^+}, \\ B_A(z_{\nu^+}, z_{\nu^+}) = \frac{1}{2y^2}, \quad B_A(z_{\nu^-}, z_{\nu^-}) = \frac{-1}{2y^2}. \end{aligned}$$

iii) We have $\mu_0 = -z' + y^2(2z_{\nu^+} - z)$.
iv) Let $\lambda \in \mathcal{O}'$ *and consider it as an element of* L'. *Then*

$$B_A(\lambda, \mu_0) = {}^t\lambda A \mu_0.$$

Proof. Part i) follows from the definition of B_A. For part ii) use $B_A(\nu, \nu) = 1$ and part i). Part iii) follows from part ii). For part iv), use $B_A(\lambda, z) = B_A(\lambda, z') = 0$. □

4.2. *The theta kernel*

Let w^+ (respectively w^-) be the orthogonal complement of the line generated by z_{ν^+} (respectively z_{ν^-}) in $\iota_\nu^+(V_5)$ (respectively $\iota_\nu^-(V_5)$). For $\lambda \in V_5$, let λ_{w^+} and λ_{w^-} be the projection of λ to w^+ and w^- respectively. We define the linear map $w : V_5 \to \mathbb{R}^{1,5}$ by $w(\lambda) = (\lambda_{w^+}, \lambda_{w^-})$, so that w is an isomorphism from w^+ and w^- to their images and w vanishes on z_{ν^+} and z_{ν^-}. For our special case, w^+ is trivial, the image of w is 4-dimensional, and the first coordinate of $w(\lambda)$ is 0.

If p is a polynomial on $\mathbb{R}^{1,5}$, we say that p has homogeneous degree (m^+, m^-) if it is homogeneous of degree m^+ in the first variable and homogeneous of degree m^- in the last 5 variables. For h^+, h^- integers satisfying $0 \le h^+ \le m^+$ and $0 \le h^- \le m^-$ define polynomials p_{w,h^+,h^-} on $w(V_5)$ of homogeneous degree $(m^+ - h^+, m^- - h^-)$ by

$$p(\iota_\nu(\lambda)) = \sum_{h^+,h^-} B_A(\lambda, z_{\nu^+})^{h^+} B_A(\lambda, z_{\nu^-})^{h^-} p_{w,h^+,h^-}(w(\lambda)). \tag{4.1}$$

Let $p : \mathbb{R}^6 \to \mathbb{R}$ be the polynomial given by $p(x_1, \cdots, x_6) = -2^{-2}x_1^2$. We get a polynomial on V_5 defined by $p \circ \iota_\nu$ given by the formula

$$p(\iota_\nu(\lambda)) = -2^{-2}B_A(\lambda, \nu)^2 = -2^{-1}y^2 B_A(\lambda, z_{\nu^+})^2.$$

By (4.1), we have

$$p_{w,h^+,h^-} = \begin{cases} -2^{-1}y^2 & \text{if } (h^+, h^-) = (2, 0); \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that the polynomial p_{w,h^+,h^-} is a constant in this case.

Let Δ be the Laplacian on $\mathbb{R}^{1,5}$. For $\tau \in \mathfrak{h}$, $(x, y) \in \mathbb{H}_5$ and $\mu \in D = L'/L$, define

$$\theta_\mu^L(\tau, \nu(x,y), p) := \sum_{\lambda \in L+\mu} \big(\exp(\frac{-\Delta}{8\pi v})(p)\big)(\iota_\nu(\lambda)) \times \exp(2\pi\sqrt{-1}\Big(Q_A(\iota_\nu^+(\lambda))\tau + Q_A(\iota_\nu^-(\lambda))\bar{\tau}\Big)),$$

$$\Theta_L(\tau, \nu(x,y), p) := \sum_{\mu \in D} e_\mu \theta_\mu^L(\tau, \nu(x,y), p).$$

Proposition 4.2. *For* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, *we have*

$$\Theta_L(\frac{a\tau + b}{c\tau + d}, \nu(x,y), p) = |c\tau + d|^5 \rho_D(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])\Theta_L(\tau, \nu(x,y), p).$$

Proof. The transformation formula in the τ variable follows from Theorem 4.1 of [3] by noticing that $b^+ = 1$, $b^- = 5$, $m^+ = 2$, and $m^- = 0$. □

4.3. *The theta lift*

Let $f \in S(\Gamma_0(N), r)$, N square-free, be an Atkin-Lehner eigenform with eigenvalues ε_c for all $c|N$. Let $\mathcal{L}_D(f)$ be the $\mathbb{C}[D]$ valued modular form as defined in (3.2). Let $\Theta_L(\tau, \nu(x, y), p)$ be the theta function defined in the previous section. Define

$$\Phi_L(\nu(x,y), p, f) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}} (\mathcal{L}_D(f))(\tau)\overline{\Theta_L(\tau, \nu(x,y), p)}v^{\frac{5}{2}}\frac{dudv}{v^2}.$$

Here, complex conjugation on $\mathbb{C}[D]$ is given by $\overline{e_\mu} := e_{-\mu}$. In the product of Θ_L and $\mathcal{L}_D(f)$, we are taking the inner product in $\mathbb{C}[D]$ to get a $\mathbb{C}$-valued function. By Propositions 3.2 and 4.2, we see that the integrand is indeed invariant under $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 4.3. *Let $\gamma \in \Gamma = \{\gamma \in \mathcal{G}(\mathbb{Q}) : \gamma L = L\}$. Then*

$$\Phi_L(\gamma\nu(x,y), p, f) = \Phi_L(\nu(x,y), p, f).$$

Proof. Using the definition of L', it is easy to see that Γ acts on L' and hence on D as well. Every element of Γ fixes 0 and, for $0 \neq \mu \in D$ with $\gamma \in \Gamma$, we have $Q_D(\mu) = Q_D(\gamma^{-1}\mu)$. Let us observe that $p(\iota_\nu(\lambda)) = -2^{-2}B_A(\lambda,\nu)^2$ and $B_A(\lambda, \gamma\nu) = B_A(\gamma^{-1}\lambda, \nu)$. Now, a change of variable gives us $\theta^L_\mu(\tau, \gamma\nu, p) = \theta^L_{\gamma^{-1}\mu}(\tau, \nu, p)$. Hence

$$\Theta_L(\tau, \gamma\nu(x,y), p) = \sum_{\mu\in D} e_\mu \theta^L_{\gamma^{-1}\mu}(\tau, \nu(x,y), p).$$

From (3.5), we know that the e_μ-component of $\mathcal{L}_D(f)$ depends only on $Q_D(\mu)$. As seen above, $Q_D(\mu) = Q_D(\gamma^{-1}\mu)$ for all $\mu \neq 0$ in D. Upon integration, we get the result. □

Let $z^\perp$ be the orthogonal complement of the line z generates in V_5. By part i) of Lemma 4.1, we see that $z \in z^\perp$. Let $K := (L \cap z^\perp)/\mathbb{Z}z$. By the definition of B_A, we can see that the lattice K is isomorphic to $\mathcal{O}$. Given $\mathcal{L}_D(f)$ as above, we can define a $\mathbb{C}[K'/K]$-valued function $(\mathcal{L}_D(f))_K$ which is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ with respect to $\rho_{K'/K}$. In our case, since $K = \mathcal{O}$, we have $K'/K = D$ and hence $(\mathcal{L}_D(f))_K = \mathcal{L}_D(f)$. We want to show that the Fourier expansion of $\Phi_L(\nu(x,y), f)$ is of the form (2.1). By Theorem 7.1 of [3], the Fourier expansion of $\Phi_L(\nu(x,y), f)$ is given by

$$
\begin{aligned}
&\Phi_L(\nu(x,y),f)\\
&\quad=\frac{1}{2\sqrt{Q_A(z_{\nu^+})}}\sum_{h\geq 0} h!\big(\frac{Q_A(z_{\nu^+})}{2\pi}\big)^h \Phi_K(w,p_{w,h,h},(\mathcal{L}_D(f))_K)\\
&\quad\quad+\frac{1}{\sqrt{Q_A(z_{\nu^+})}}\sum_{h\geq 0}\sum_{h^+,h^-}\frac{h!\big(\frac{-2Q_A(z_{\nu^+})}{\pi}\big)^h}{(2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}\\
&\quad\quad\times\sum_j\sum_{\lambda\in K'}\frac{(-\Delta)^j(\bar{p}_{w,h^+,h^-})(w(\lambda))}{(8\pi)^j j!}\\
&\quad\quad\times\sum_{n>0} e(B_A(n\lambda,\mu_0))n^{h^++h^--2h}\sum_{\substack{\mu\in D\\ \mu|_{L\cap z^\perp}=\lambda}} e(nB_A(\mu,z'))\\
&\quad\quad\times\int_{v>0} c_{\mu,Q_A(\lambda)}(v)\exp(-\frac{\pi n^2}{4vQ_A(z_{\nu^+})}-2\pi v(Q_A(\lambda_{w^+})-Q_A(\lambda_{w^-})))\\
&\quad\quad\quad\times v^{h-h^+-h^--j-\frac{5}{2}}dv.
\end{aligned}
$$

Here

$$
v^{\frac{5}{2}}(\mathcal{L}_D(f))(u+iv)=\sum_{\mu\in D} e_\mu\sum_{m\in\mathbb{Q}} c_{\mu,m}(v)e(mu).
$$

Also, μ_0 is as defined in part iii) of Lemma 4.1. In addition, λ_{w^+} and λ_{w^-} are defined in the beginning of Section 4.2. Let us now apply this formula to our particular situation.

i) We have $K=\mathcal{O}$, hence $K'=\mathcal{O}'$.

ii) By (4.2), we have

$$
p_{w,h^+,h^-}=\begin{cases}-2^{-1}y^2 & \text{if } (h^+,h^-)=(2,0);\\ 0 & \text{otherwise.}\end{cases}
$$

Hence, the first sum is zero. In the second sum over h, h^+, and h^-, we only have the case $h=0$, $h^+=2$, and $h^-=0$. Since p_{w,h^+,h^-} is a constant function, the sum over j vanishes for all $j>0$. Hence, we only get $j=0$.

iii) By Lemma 4.1, we have $Q_A(z_{\nu^+})=(4y^2)^{-1}$ and $B_A(\lambda,\mu_0)={}^t\lambda A\mu_0$.

iv) Since $D=\mathcal{O}'/\mathcal{O}$, we can see that, for $\mu\in D$, we have $B_A(\mu,z')\in\mathbb{Z}$. Hence, $e(nB_A(\mu,z'))=1$ for all $\mu\in D$. Furthermore, for each λ, there is exactly one $\mu\in D$ such that $\mu|_{L\cap z^\perp}=\lambda$.

v) For $\lambda\in\mathcal{O}'$, we have $Q_A(\lambda)=-Q_{A_0}(\lambda)$.

vi) For $\lambda\in\mathcal{O}'$, we have $\lambda_{w^+}=0$ since w^+ is the trivial space in our case. We have $Q_A(\lambda_{w^-})=-Q_{A_0}(\lambda)$.

vii) By (3.5), for $\mu\in D$, $\lambda\in\mathcal{O}'$, and $v\in\mathbb{R}_{>0}$, we have

$$
c_{\mu,Q_A(\lambda)}(v)=c_\mu(\lambda)W_{0,\frac{\sqrt{-1}r}{2}}(4\pi Q_{A_0}(\lambda)v)v^{\frac{5}{2}},
$$

where

$$c_\mu(\lambda) = \begin{cases} 0 & \text{if } \lambda \notin \mu + \mathcal{O}; \\ \sum_{c \mid \frac{N}{q_\mu}} \prod_{p \mid \frac{N}{c}} (-\varepsilon_p) c(-Q_{A_0}(\lambda)\frac{N}{c}) & \text{if } \lambda \in \mu + \mathcal{O}. \end{cases} \tag{4.3}$$

Let $\mu_\lambda \in D$ be such that $\lambda \in \mu_\lambda + \mathcal{O}$. Then we see that $c_\mu(\lambda)$ is non-zero only if $\mu = \mu_\lambda$.

viii) We have

$$\begin{aligned} &\int_{v>0} v^{\frac{5}{2}} W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|Q_{A_0}(\lambda)|v)\exp(-\frac{\pi n^2 y^2}{v} - 2\pi v Q_{A_0}(\lambda))v^{-2-\frac{5}{2}}dv \\ =&\int_{v>0} W_{0,\frac{\sqrt{-1}r}{2}}(4\pi\frac{|Q_{A_0}(\lambda)|}{v})\exp(-\pi n^2y^2v - \frac{2\pi Q_{A_0}(\lambda)}{v})dv \quad \text{by } (v \mapsto \frac{1}{v}) \\ =&4\frac{\sqrt{Q_{A_0}(\lambda)}}{ny}K_{\sqrt{-1}r}(4\pi\sqrt{Q_{A_0}(\lambda)}ny). \end{aligned}$$

We have used

$$\int_0^\infty \exp(-pt - \frac{a}{2t})W_{0,\frac{\sqrt{-1}r}{2}}(\frac{a}{t})dt = 2\sqrt{\frac{a}{p}}K_{\sqrt{-1}r}(2\sqrt{ap})$$

(cf. [4, 4.22 (22)]) with $a = 4\pi Q_{A_0}(\lambda)$ and $p = \pi n^2 y^2$.

Putting this together, we see that the Fourier expansion is given by

$$\begin{aligned} &2y(\frac{-1}{4})\sum_{\lambda\in\mathcal{O}'}\frac{-y^2}{2}\sum_{n>0}e(n^t\lambda A_0 x)n^2 c_{\mu_\lambda}(\lambda)4\frac{\sqrt{Q_{A_0}(\lambda)}}{ny}K_{\sqrt{-1}r}(4\pi\sqrt{Q_{A_0}(\lambda)}ny) \\ &= \sum_{\lambda\in\mathcal{O}'}\sum_{n>0} n\sqrt{Q_{A_0}(\lambda)}c_{\mu_\lambda}(\lambda)y^2K_{\sqrt{-1}r}(4\pi\sqrt{Q_{A_0}(\lambda)}ny)e(n^t\lambda A_0 x) \\ &= \sum_{\beta\in\mathcal{O}'}\sqrt{Q_{A_0}(\beta)}\Big(\sum_{\substack{d>0 \\ \frac{1}{d}\beta\in\mathcal{O}'}} c_{\mu_{\frac{\beta}{d}}}(\frac{\beta}{d})\Big)y^2K_{\sqrt{-1}r}(4\pi\sqrt{Q_{A_0}(\beta)}y)e({}^t\beta A_0 x). \end{aligned}$$

From Lemma 4.3 and the above Fourier expansion (compare to (2.1)), we get

Theorem 4.4. *$\Phi_L(\nu(x,y),f)$ belongs to $\mathcal{M}(\Gamma, \sqrt{-1}r)$.*

We will use the remaining section to obtain a formula for the Fourier coefficients of $\Phi_L(\nu(x,y),f)$ in terms of the Fourier coefficients of f. For $\beta \in \mathcal{O}'$, set

$$A(\beta) = \sqrt{Q_{A_0}(\beta)}\sum_{\substack{d>0 \\ \frac{1}{d}\beta\in\mathcal{O}'}} c_{\mu_{\frac{\beta}{d}}}(\frac{\beta}{d}). \tag{4.4}$$

We will now obtain a formula for $A(\beta)$ in terms of the Fourier coefficients $c(n)$ of f. Let us define the primitive elements of $\mathcal{O}'$ by

$$\mathcal{O}'_{\text{prim}} := \{\beta \in \mathcal{O}' : \frac{1}{n}\beta \notin \mathcal{O}' \text{ for all positive integers } n > 1\}.$$

Proposition 4.5. *Write* $\beta \in \mathcal{O}'$ *as*

$$\beta = \prod_{p|N} p^{u_p} n\beta_0, \qquad u_p \geq 0, n > 0, \gcd(n, N) = 1 \text{ and } \beta_0 \in \mathcal{O}'_{\text{prim}}.$$

Let $q_{\beta_0} = q_{\mu_{\beta_0}}$. *For* $p|N$, *set*

$$\delta_p = \begin{cases} 0 & \text{if } p|q_{\beta_0}; \\ 1 & \text{if } p \nmid q_{\beta_0}. \end{cases}$$

Then

$$A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{p|N} \sum_{t_p=0}^{2u_p+\delta_p} \sum_{d|n} c\Big(\frac{-Q_{A_0}(\beta)}{\prod\limits_{p|N} p^{t_p-1} d^2}\Big) \prod_{p|N} (-\varepsilon_p)^{t_p-1}. \tag{4.5}$$

Proof. From (4.3) and (4.4), it is clear that we can take $n = 1$ above. Let S_0 be the set of primes dividing q_{β_0} and S' be the subset of S_0 with $u_p = 0$. For any set of primes S, denote by N_S the product of all primes in S. From (4.3) and (4.4), we have

$$\begin{aligned} A(\beta) =& \sqrt{Q_{A_0}(\beta)} \sum_{\substack{p|N \\ a_p=0}}^{u_p} c_{\mu_{\beta/(\prod_{p|N} p^{a_p})}} \Big(\frac{\beta}{\prod_{p|N} p^{a_p}}\Big) \\ =& \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{\substack{p|(N/N_{S_0}) \\ a_p=0}}^{u_p} \sum_{\substack{p|(N_{S_0}/N_S) \\ a_p=0}}^{u_p-1} c_{\mu_{\beta/(\prod_{p|N_S} p^{u_p} \prod_{p|(N/N_S)} p^{a_p})}} \\ & \times \Big(\frac{\beta}{\prod_{p|N_S} p^{u_p} \prod_{p|(N/N_S)} p^{a_p}}\Big). \end{aligned}$$

We are essentially splitting up the sum according to which $a_p = u_p$ for $p \in S_0$ so that, for all the β' appearing in the sum above, we have $q_{\beta'} = N_S$. Hence applying (4.3), we have

$$\begin{aligned} A(\beta) =& \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{\substack{p|(N/N_{S_0}) \\ a_p=0}}^{u_p} \sum_{\substack{p|(N_{S_0}/N_S) \\ a_p=0}}^{u_p-1} \sum_{c|(N/N_S)} \\ & \times \prod_{p|\frac{N}{c}} (-\varepsilon_p) c\Big(-Q_{A_0}\Big(\frac{\beta}{\prod_{p|N_S} p^{u_p} \prod_{p|(N/N_S)} p^{a_p}}\Big)\frac{N}{c}\Big). \end{aligned}$$

Here, if $p|N_S$ then we have $p|(N/c)$ for all $c|(N/N_S)$. Hence, we have

$$\begin{aligned} A(\beta) &= \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{\substack{p|(N/N_{S_0}) \\ a_p=0}}^{u_p} \sum_{\substack{p|(N_{S_0}/N_S) \\ a_p=0}}^{u_p-1} \sum_{c|(N/N_S)} \prod_{p|N_S} (-\varepsilon_p) \\ &\quad \times \prod_{p|\frac{N}{cN_S}} (-\varepsilon_p) c\Big(\frac{-Q_{A_0}(\beta)}{\prod_{p|N_S} p^{2u_p-1} \prod_{p|\frac{N}{cN_S}} p^{2a_p-1} \prod_{p|c} p^{2a_p}} \Big) \\ &= \sqrt{Q_{A_0}(\beta)} \sum_{S' \subset S \subset S_0} \sum_{\substack{p|(N/N_{S_0}) \\ a_p=0}}^{u_p} \sum_{\substack{p|(N_{S_0}/N_S) \\ a_p=0}}^{u_p-1} \sum_{c|(N/N_S)} \prod_{p|N_S} (-\varepsilon_p) \\ &\quad \times \prod_{p|\frac{N}{cN_S}} (-\varepsilon_p)^{2a_p-1} \prod_{p|c} (-\varepsilon_p)^{2a_p} c\Big(\frac{-Q_{A_0}(\beta)}{\prod_{p|N_S} p^{2u_p-1} \prod_{p|\frac{N}{cN_S}} p^{2a_p-1} \prod_{p|c} p^{2a_p}} \Big). \end{aligned}$$

Further, we can divide the set $\{c|(N/N_S)\}$ into a set of pairs (c, c') such that $cc' = N/N_S$. Note that for any prime $p|(N/N_S)$ and any pair (c, c') as above, p divides exactly one of N/c or N/c'. Also note, in the denominator in the last term above, the possible exponents of p are as follows: if $p|q_{\beta_0}$, then the exponents vary from 0 to $2u_p - 1$, and if $p \nmid q_{\beta_0}$, then the exponents vary from 0 to $2u_p$. In addition, a term for $-\varepsilon_p$ appears in the product if and only if the exponent of p in the denominator is odd. Changing variable from a_p to t_p, and noting the definition of δ_p in the statement of the proposition, we finally get the formula in (4.5). □

5. The cuspidality of the theta lifts

We show the cuspidality of our theta lifts

$$\Phi_L(\nu(x,y), p, f) := \int_{SL_2(\mathbb{Z})\backslash\mathfrak{h}} \mathcal{L}_D(f) \overline{\Theta_L(\tau, \nu(x,y), p)} v^{\frac{5}{2}} \frac{dudv}{v^2}.$$

The first step is to understand the action of $c \in \mathcal{G}(\mathbb{Q})$ on $\Phi_L(\nu(x,y), p, f)$.

Lemma 5.1. *For any $c \in \mathcal{G}(\mathbb{Q})$, we have*

$$\Phi_L(c\nu(x,y), p, f) = \Phi_{c^{-1}L}(\nu(x,y), p, f).$$

Proof. Recall that $\iota_{g\cdot\nu}(g\cdot\lambda) = g\cdot\iota_\nu(\lambda)$ for $(g, \lambda, \nu) \in \mathcal{G}(\mathbb{R}) \times \mathbb{R}^6 \times \mathcal{D}^+$, which yields

$$\iota^+_{g\cdot\nu}(g\cdot\lambda) = g\cdot\iota^+_\nu(\lambda),\ \iota^-_{g\cdot\nu}(g\cdot\lambda) = g\cdot\iota^-_\nu(\lambda).$$

In addition, we note that

$$p(\iota_{g\cdot\nu}(\lambda) = p(\iota_\nu(g^{-1}\cdot\lambda))$$

for $(g, \lambda) \in \mathcal{G}(\mathbb{R}) \times \mathbb{R}^6$. For $(g, \mu) \in \mathcal{G}(\mathbb{R}) \times L'$ we thereby have $\theta_\mu^L(\tau, g \cdot \nu(x, y), p) =$

$$\begin{aligned}
&\sum_{\lambda \in \mu + L} \exp((\frac{-\Delta}{8\pi y}(p))(\iota_{g\cdot\nu}(\lambda))\mathbf{e}(Q(\iota_{g\cdot\nu}^+(\lambda))\tau + Q(\iota_{g\cdot\nu}^-(\lambda))\bar{\tau}) \\
=&\sum_{\lambda \in \mu + L} \exp((\frac{-\Delta}{8\pi y}(p))(\iota_\nu(g^{-1} \cdot \lambda))\mathbf{e}(Q(g \cdot \iota_\nu^+(g^{-1} \cdot \lambda))\tau + Q(g \cdot \iota_\nu^-(g^{-1} \cdot \lambda))\bar{\tau}) \\
=&\sum_{\lambda \in g^{-1}\cdot(\mu + L)} \exp((\frac{-\Delta}{8\pi y}(p))(\iota_\nu(\lambda))\mathbf{e}(Q(\iota_\nu^+(\lambda))\tau + Q(\iota_\nu^-(\lambda))\bar{\tau}) \\
=&\theta_{g^{-1}\mu}^{g^{-1}L}(\tau, \cdot\nu(x, y), p).
\end{aligned}$$

Writing $\mathcal{L}_D(f) = \sum_{\mu \in D} f_\mu^D e_\mu$ and using (3.6), we get

$$\begin{aligned}
\mathcal{L}_D(f)\overline{\Theta_L(\tau, c\nu, p)} &= \sum_{\mu \in D} f_\mu^D \overline{\theta_{-c^{-1}\mu}^{c^{-1}L}(\tau, \cdot\nu(x, y), p)} \\
&= \sum_{\mu \in D} f_{c^{-1}\mu}^{c^{-1}D} \overline{\theta_{-c^{-1}\mu}^{c^{-1}L}(\tau, \cdot\nu(x, y), p)} \\
&= \sum_{\mu \in c^{-1}D} f_\mu^{c^{-1}D} \overline{\theta_{-\mu}^{c^{-1}L}(\tau, \cdot\nu(x, y), p)} \\
&= \mathcal{L}_{c^{-1}D}(f)\overline{\Theta_{c^{-1}L}(\tau, \nu, p)}.
\end{aligned}$$

Here we put $c^{-1}D := c^{-1}L'/c^{-1}L$, which is isomorphic to D. Upon integration, we get the result of the lemma. □

For any cusp $c \in \mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q})/\Gamma$, we see that $Q_A(x) = Q_A(c^{-1}x)$ for all $x \in \mathbb{Q}^6$. Hence, the lattice $c^{-1}L$ has the same associated quadratic form as L. Therefore, the discriminant form $c^{-1}D = c^{-1}L'/c^{-1}L$ is isomorphic to $D = L'/L$ as quadratic modules and hence Proposition 4.5 applies to the Fourier expansion of $\Phi_{c^{-1}L}(\nu(x, y), p, f)$. In particular, $\Phi_{c^{-1}L}(\nu(x, y), p, f)$ has no constant term and therefore we get the following.

Proposition 5.2. *For each representative* c *of the* Γ*-cusps,* $\Phi_L(c\nu(x, y), p, f)$ *has no constant term. Namely, our lifts* $\Phi_L(\nu(x, y), p, f)$ *are cuspidal.*

6. Hecke Theory

6.1. *Adelization of automorphic forms*

To study the action of the Hecke operators on our cusp forms constructed by the lift, we need the adelic as well as non-adelic treatment of automorphic forms.

For $h \in \mathcal{H}(\mathbb{A})$, we have the decomposition $h = au^{-1}$ with $(a, u) \in \mathrm{GL}_4(\mathbb{Q}) \times (\Pi_{p<\infty}\mathrm{SL}_4(\mathbb{Z}_p) \times \mathrm{SL}_4(\mathbb{R}))$. Let $\mathcal{O}_h := (\Pi_{p<\infty} h_p \mathbb{Z}_p^4 \times \mathbb{R}^4) \cap \mathbb{Q}^4$ for $h = (h_v)_{v \leq \infty} \in \mathcal{H}(\mathbb{A})$. Then, we have $\mathcal{O}_h = a\mathcal{O}$ (c.f. [21, Section 3.3]). The dual lattice $\mathcal{O}_h'$ is then equal to $a^{-1}\mathcal{O}'$. Here note that we regard $\mathcal{O}$ and $\mathcal{O}'$ as $\mathbb{Z}^4$ equipped with the quadratic forms induced by the reduced norm.

To obtain an adelic Fourier expansion, let $f \in S(\Gamma_0(N), r)$ be a Maass cusp form with the Fourier expansion $f(z) = \sum_{n\neq 0} c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|y)e(x)$. Let Λ be the standard additive character of $\mathbb{A}/\mathbb{Q}$. We introduce the following Fourier series

$$F_f(n(x)a_y kg) := \sum_{\lambda\in\mathbb{Q}^4\setminus\{0\}} F_{f,\lambda}(n(x)a_y kg) \quad \forall(x,y,k,g) \in \mathbb{A}^4 \times \mathbb{R}_+^\times \times K_\infty \times \mathcal{G}(\mathbb{A}_f) \tag{6.1}$$

with

$$F_{f,\lambda}(n(x)a_y kg) := A_\lambda(g) y^2 K_{\sqrt{-1}r}(4\pi|\lambda|_A y)\Lambda({}^t\lambda A x),$$

where $A_\lambda(g)$ is defined by the following conditions:

$$A_\lambda\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right) := \begin{cases} \sqrt{Q_{A_0}(\lambda)} \sum\limits_{p|N} \sum\limits_{t_p=0}^{2u_p+\delta_p} \sum\limits_{d|n} c\Big(\frac{-Q_{A_0}(\lambda)}{\prod_{p|N} p^{t_p-1}d^2}\Big) \prod\limits_{p|N}(-\varepsilon_p)^{t_p-1} & (\lambda \in \mathcal{O}'_h) \\ 0 & (\lambda \in \mathbb{Q}^4 \setminus \mathcal{O}'_h) \end{cases}$$

$$A_\lambda\left(\begin{pmatrix}s & & \\ & h & \\ & & s^{-1}\end{pmatrix}\right) := ||s||_{\mathbb{A}}^2 A_{||s||_{\mathbb{A}}^{-1}\lambda}\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right)$$

$$A_\lambda(n(x)gk) := \Lambda({}^t\lambda A x)A_\lambda(g) \qquad \forall(x,g,k) \in \mathbb{A}_f^4 \times \mathcal{G}(\mathbb{A}_f) \times K_f.$$

Here

1. u_p, δ_p and n are as defined in Proposition 4.5 for $\beta = h^{-1}\lambda$.
2. $(s,h) \in \mathbb{A}_f^\times \times \mathcal{H}(\mathbb{A}_f)$ and $||s||_{\mathbb{A}}$ denotes the idele norm of s.

Note, the definitions of u_p, δ_p and n do not depend on the decomposition $h = au^{-1}$, which was essentially pointed out in the proof of [21, Lemma 3.2]. The following lemma is settled by the same reasoning as [21, Lemma 3.2].

For $r \in \mathbb{C}$, let $\mathcal{M}(\mathcal{G}(\mathbb{A}), r)$ denote the space of smooth functions F on $\mathcal{G}(\mathbb{A})$ satisfying the following conditions:

1. $\Omega \cdot F = \frac{1}{8}(r^2 - 4)F$, where Ω is the Casimir operator defined in [21].
2. For any $(\gamma, g, k) = \mathcal{G}(\mathbb{Q}) \times \mathcal{G}(\mathbb{A}) \times K$, we have $F(\gamma g k) = F(g)$.
3. F is of moderate growth.

Note that $F \in \mathcal{M}(\mathcal{G}(\mathbb{A}), r)$ has the Fourier expansion

$$F(g) = \sum_{\lambda\in\mathbb{Q}^4} F_\lambda(q), \quad F_\lambda(g) := \int_{\mathbb{A}^4/\mathbb{Q}^4} F(n(x)g)\Lambda({}^t\lambda A x)dx,$$

where dx is the invariant measure normalized so that the volume of $\mathbb{A}^4/\mathbb{Q}^4$ is one. The adelic function F is called a cusp form if $F_0 \equiv 0$ in the Fourier expansion.

Proposition 6.1. *The adelic function F_f is a cusp form belonging to $\mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$.*

Proof. By the argument similar to [21, Theorem 3.3] this follows from the Fourier expansion discussed in Section 4.3. □

6.2. *Sugano Theory*

We will show that if f is a Hecke eigenform then F_f is an Hecke eigenform by using the non-archimedean local theory of Sugano [38, Section 7]. For a prime p, let $F = \mathbb{Q}_p$ with the ring of integers $\mathbb{Z}_p$. Let $n_0 \le 4$ and let $S_0 \in M_{n_0}(F)$ be an anisotropic even symmetric matrix of degree n_0. For the $m \times m$ matrix $J_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$, let G_m denote the group of F-valued points of the orthogonal group of degree $2m + n_0$, defined by the matrix $Q = \begin{pmatrix} & & J_m \\ & S_0 & \\ J_m & & \end{pmatrix}$. Denote by $L_m := \mathbb{Z}_p^{2m+n_0}$ the maximal lattice with respect to Q_m and let K_m be the maximal compact open subgroup of G_m defined by the lattice

$$K_m := \{g \in G_m \mid gL_m = L_m\}. \tag{6.2}$$

Let $\mathcal{H}_m$ be Hecke algebra for (G_m, K_m) and define $C_m^{(r)} \in \mathcal{H}_m$ to be the double cosets $K_m c_m^{(r)} K_m$, where

$$c_m^{(r)} := \mathrm{diag}(p, \ldots, p, 1, \ldots 1, p^{-1}, \ldots, p^{-1}) \in G_m$$

which is a diagonal matrix whose first r and last r entries are p and p^{-1} respectively. By [38, Section 7], $\{C_m^{(r)} \mid 1 \le r \le m\}$ forms generators of the Hecke algebra $\mathcal{H}_m$.

We embed G_i for $i \le m$ in G_m as a subgroup by the middle $(2i+n_0) \times (2i+n_0)$ block. We regard K_i as a subgroup of K_m similarly. The invariant measure of G_m is normalized so that the volume of K_i is one for each $i \le m$.

For a prime $p \nmid N$, we have $n_0 = 0$ and $m = 3$. In this case, the lattice L_3 is self-dual. For a non-negative integer k, let

$$f_{k,j} := \frac{p^{j-1}(p^{k-j+1} - 1)(p^{k-j} + 1)}{p^j - 1} \quad (\forall j \in \mathbb{Z} \setminus \{0\}), \tag{6.3}$$

a special case of [38, 7.11] for $n_0 = \delta = 0$. For positive integers k, r, set $R_k^{(r)} := K_k/(K_k \cap c_k^{(r)} K_k (c_k^{(r)})^{-1})$, and let $|R_k^{(r)}|$ denote the cardinality of $R_k^{(r)}$. We have

$$|R_k^{(r)}| = \begin{cases} \Pi_{j=1}^r f_{k,j} & (1 \le r \le k); \\ 1 & (r = 0). \end{cases} \tag{6.4}$$

Following the methods in Section 4 of [21], we get the following theorem (essentially Theorem 4.11 of [21] for $n = 1/2$).

Theorem 6.2. *Suppose that f is a Hecke eigenform and let λ_p be the Hecke eigenvalue of f at $p < \infty$ with $p \nmid N$. Then the following holds.*

i) F_f is a Hecke eigenform.
ii) Let μ_i be the Hecke eigenvalue with respect to the Hecke operator $C_3^{(i)}$ for $1 \le i \le 3$. We have

$$\mu_1 = p^2(\lambda_p^2 - 2) + pf_{2,1} = p^2(\lambda_p^2 + p + p^{-1});$$

$$\mu_i = |R_2^{(i-1)}| \left(\mu_1 - \frac{p^{i-1}-1}{p^i - 1} f_{3,1} \right), (i = 2, 3).$$

6.3. *The case p | N*

When $p \mid N$, we have $m = 1$ and $n_0 = 4$. Hence, the Hecke algebra $\mathcal{H}_1$ is generated by $C_1^{(1)}$ which is the double coset $K_1 c_1^{(1)} K_1$ as defined in Section 6.2. Let $n(x) \in G_1$ be as defined in Section 2.1 and let $(t, g) := \mathrm{diag}(t, g, t^{-1}) \in G_1$ for $t \in \mathbb{Q}_p^\times$ and $g \in G_0$.

Lemma 6.3.

$$C_1^{(1)} = \bigsqcup_{x \in \mathfrak{X}_1} (p, 1_4) n(x) K_1 \sqcup \bigsqcup_{x \in \mathfrak{X}_3} (1, 1_4) n(x) K_1 \sqcup (p^{-1}, 1_4) K_1$$

where

$$\mathfrak{X}_1 = \left\{ x \in p^{-1}\mathcal{O}/\mathcal{O} \right\}, \, \mathfrak{X}_3 = \{ x \in (\mathcal{O}' - \mathcal{O})/\mathcal{O} \}.$$

Proof. This is a direct result of [38, Lemma 7.1] for $v = 0$ and $r = 1$. Note, $c_0^{(0)} = 1_4$ and hence, $R_0^{(0)} = 1$ and $u = 1_4$. $\mathfrak{X}_{0,1}^{(1)}$ and $\mathfrak{X}_{0,3}^{(1)}$ simplify as above whereas $\mathfrak{X}_{0,2}^{(1)}$ and $\mathfrak{X}_{0,4}^{(1)}$ are empty since $r = 1$ and $v = 0$ respectively. □

We can now describe the action of $C_1^{(1)}$ with the invariant measure dx of G_1 normalized so that the volume $\int_{K_1} dx = 1$. Define

$$(C_1^{(1)} \cdot \Phi)(g) := \int_{G_1} \mathrm{char}_{K_1 c_1^{(1)} K_1}(x) \Phi(gx) dx$$

for $\Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}), r)$. The following proposition derives the action of $C_1^{(1)}$ on Fourier coefficients of Φ.

Proposition 6.4. *Let* $\Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$ *be a lift. Then*

$$(C_1^{(1)} \cdot \Phi)(n(x)a_y) = \sum_{\lambda \in \mathcal{O}' \setminus \{0\}} A'_\lambda(1) y^2 K_{\sqrt{-1}r}(4\pi \sqrt{Q_{A_0}(\lambda)} y) \Lambda({}^t\lambda A_0(x)),$$

where

$$A'_\lambda(1) = \begin{cases} p^2 A_{p\lambda}(1) - A_\lambda(1) + p^2 A_\lambda(1) + p^2 A_{p^{-1}\lambda}(1) & \text{if } \lambda \in p\mathcal{O}' \setminus \{0\}; \\ p^2 A_{p\lambda}(1) - A_\lambda(1) + p^2 A_\lambda(1) & \text{if } \lambda \in \mathcal{O} \setminus p\mathcal{O}'; \\ p^2 A_{p\lambda}(1) - A_\lambda(1) & \text{if } \lambda \in \mathcal{O}' \setminus \mathcal{O}. \end{cases}$$

Proof. Since $\int_{K_1} dx = 1$, Lemma 6.3 implies that the action of $C_1^{(1)}$ on Φ can be expressed as

$$(C_1^{(1)} \cdot \Phi)(g) = \sum_{x \in \mathfrak{X}_1} \Phi(g(p, 1_4)n(x)) + \sum_{x \in \mathfrak{X}_3} \Phi(gn(x)) + \Phi(g(p^{-1}, 1_4)).$$

Here, we are using the fact that $\Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$ is right invariant under K_1. Let $g = n(x_0)a_y$ with $x_0 \in \mathbb{A}^4$ and $y \in \mathbb{R}_+$. Let $a_p^{\#} := \mathrm{diag}(p, 1_4, p^{-1})$ embedded diagonally in $\mathcal{G}(\mathbb{Q})$. We will abuse the notation to denote $(1_\infty, \dots, (p, 1_4), \dots)$ and $(1_\infty, \dots, n(x), \dots)$ by $(p, 1_4)$ and $n(x)$ respectively, where the nontrivial terms are at the p-th place. Hence,

$$\begin{aligned}(C_1^{(1)} \cdot \Phi)(n(x_0)a_y) = &\sum_{x \in \mathfrak{X}_1} \Phi(n(x_0)a_y(p, 1_4)n(x)) \\ &+ \sum_{x \in \mathfrak{X}_3} \Phi(n(x_0)a_y n(x)) + \Phi(n(x_0)a_y(p^{-1}, 1_4)).\end{aligned}$$

Note,

$$\begin{aligned}&\Phi(n(x_0)a_y(p, 1_4)n(x)) \\ &\qquad = \Phi(a_{p^{-1}}^{\#} n(x_0)a_y(p, 1_4)n(x)) \\ &\qquad = \Phi(n(p^{-1}x_0)a_{p^{-1}y}(1_\infty, (p^{-1}, 1_4), \dots, (1, 1_4), (p^{-1}, 1_4), \dots)n(x)) \\ &\qquad = \Phi(n(p^{-1}x_0)a_{p^{-1}y}n(x)(1_\infty, (p^{-1}, 1_4), \dots, (1, 1_4), (p^{-1}, 1_4), \dots)) \\ &\qquad = \Phi(n(p^{-1}x_0)n(x)a_{p^{-1}y}).\end{aligned}$$

We obtain the last equality as $n(x)$ and $a_{p^{-1}y}$ commute, and $(1_\infty, (p^{-1}, 1_4), \dots, (1, 1_4), (p^{-1}, 1_4), \dots)$ belongs to the maximal compact $K_f K_\infty$. By similar computation for other terms, we obtain

$$\begin{aligned}&(C_1^{(1)} \cdot \Phi)(n(x_0)a_y) \\ &\quad = \sum_{x \in \mathfrak{X}_1} \Phi(n(p^{-1}x_0)n(x)a_{p^{-1}y}) + \sum_{x \in \mathfrak{X}_3} \Phi(n(x_0)n(x)a_y) + \Phi(n(px_0)a_{py}) \\ &\quad = \sum_{\mathbb{Q}^4 \setminus \{0\}} A_\lambda(1)(p^{-1}y)^2 K_{ir}(4\pi\sqrt{Q_{A_0}(\lambda)}p^{-1}y) \sum_{x \in \mathfrak{X}_1} \Lambda({}^t\lambda A_0(p^{-1}x_0)_{p,x}) \\ &\quad\quad + \sum_{\mathbb{Q}^4 \setminus \{0\}} A_\lambda(1)y^2 K_{ir}(4\pi\sqrt{Q_{A_0}(\lambda)}y) \sum_{x \in \mathfrak{X}_3} \Lambda({}^t\lambda A_0(x_0)_{p,x}) \\ &\quad\quad + \sum_{\mathbb{Q}^4 \setminus \{0\}} A_\lambda(1)(py)^2 K_{ir}(4\pi\sqrt{Q_{A_0}(\lambda)}py)\Lambda({}^t\lambda A_0(px_0)).\end{aligned} \tag{6.5}$$

Here, $(p^{-1}x_0)_{p,x}$ is $p^{-1}x_{0,v}$ at all places $v \neq p$ and is $p^{-1}x_{0,p} + x$ at the place p. Similarly, $(x_0)_{p,x}$ is $x_{0,v}$ at all places $v \neq p$ and $x_{0,p} + x$ at the place p. Note,

$$\sum_{x \in \mathfrak{X}_1} \Lambda({}^t\lambda A_0(p^{-1}x_0)_{p,x}) = \Lambda({}^t\lambda A_0(p^{-1}x_0)) \sum_{x \in \mathfrak{X}_1} \Lambda({}^t\lambda A_0 x) \tag{6.6}$$

with the summation over $x \in \mathfrak{X}_1$ happening only at the p-th place. As Λ is an additive character being summed over a group $\mathfrak{X}_1 = \left\{x \in p^{-1}\mathcal{O}/\mathcal{O}\right\}$, we get

$$\sum_{x \in \mathfrak{X}_1} \Lambda({}^t\lambda A_0 x) = \begin{cases} p^4 & p^{-1}\lambda \in \mathcal{O}'; \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

Similarly,

$$\sum_{x\in\mathfrak{X}_3}\Lambda({}^t\lambda A_0((x_0)_{p,x}))=\Lambda({}^t\lambda A_0(x_0))\sum_{x\in\mathfrak{X}_3}\Lambda({}^t\lambda A_0x) \tag{6.8}$$

being summed over $\mathfrak{X}_3=\{x\in(\mathcal{O}'-\mathcal{O})/\mathcal{O}\}$. Note,

$$\sum_{x\in\mathfrak{X}_3}\Lambda({}^t\lambda A_0x)=\sum_{x\in\mathcal{O}'/\mathcal{O}}\Lambda({}^t\lambda A_0x)-1.$$

Hence, using that Λ is an additive character being summed over a group $\mathcal{O}'/\mathcal{O}$, we get

$$\sum_{x\in\mathfrak{X}_3}\Lambda({}^t\lambda A_0x)=\begin{cases}p^2-1 & \lambda\in\mathcal{O};\\ -1 & \text{otherwise.}\end{cases} \tag{6.9}$$

Therefore, substituting (6.6)–(6.9) in (6.5) we get the formula for $A'_\lambda(1)$ as defined in the statement of the proposition. □

To write the action of the Hecke operator in terms of Fourier coefficients given in Proposition 4.5, we write $A_\lambda(1)=A(\beta)$ where $\beta=\prod_{p|N}p^{u_p}n\beta_0$ as in the proposition. Note, for $\lambda\in\mathcal{O}'$ and $\beta\in\mathcal{O}'$ the conditions for $A'_\lambda(1)$ on λ from Proposition 6.4 above translate to conditions on β as follows:

$$\begin{aligned}\lambda\in p\mathcal{O}'\setminus\{0\} &\iff u_p\geq 1;\\ \lambda\in\mathcal{O}\setminus p\mathcal{O}' &\iff u_p=0,\delta_p=1;\\ \lambda\in\mathcal{O}'\setminus\mathcal{O} &\iff u_p=0,\delta_p=0.\end{aligned}$$

Then, as

$$A_{p\lambda}(1)=A(p\beta);\quad A_{p^{-1}\lambda}(1)=A(p^{-1}\beta)$$

we can rewrite the $A'_\lambda(1)$ in terms of β as

$$A'_\lambda(1)=\begin{cases}p^2A(p\beta)+(p^2-1)A(\beta)+p^2A(p^{-1}\beta) & \text{if } u_p\geq 1;\\ p^2A(p\beta)+(p^2-1)A(\beta) & \text{if } u_p=0,\delta_p=1;\\ p^2A(p\beta)-A(\beta) & \text{if } u_p=0,\delta_p=0.\end{cases} \tag{6.10}$$

Let $f\in S(\Gamma_0(N),r)$ be a newform with Hecke eigenvalue λ_p for the operator defined by the action of the double coset $\Gamma_0(N)\left[\begin{smallmatrix}1&\\&p\end{smallmatrix}\right]\Gamma_0(N)$ at prime p. Assuming it is an Atkin Lehner eigenform with eigenvalue ϵ_p, it can be shown that

$$\lambda_p=-\epsilon_p. \tag{6.11}$$

Using the single coset decomposition

$$\Gamma_0(N)\begin{bmatrix}1&\\&p\end{bmatrix}\Gamma_0(N)=\bigsqcup_{b=0}^{p-1}\Gamma_0(N)\begin{bmatrix}1&b\\&p\end{bmatrix}$$

([19, Lemma 9.14]) we have

$$\sum_{b=0}^{p-1} f(\frac{z+b}{p}) = \lambda_p f(z).$$

In terms of Fourier coefficients, using (6.11), we get

$$c(pm) = \frac{\lambda_p}{p} c(m) = \frac{-\epsilon_p}{p} c(m) \quad \forall m \in \mathbb{Z}.$$

Therefore,

$$c(m) = \frac{p}{-\varepsilon_p} c(pm) \quad \forall m \in \mathbb{Z},$$

and

$$\begin{aligned} c(\frac{-Q_{A_0}(\beta)}{p^{t_p-1}\prod\limits_{\substack{\ell|N\\ \ell\neq p}} \ell^{t_\ell-1} d^2}) \prod_{\substack{\ell|N\\ \ell\neq p}} (-\varepsilon_\ell)^{t_\ell-1}(-\varepsilon_p)^{t_p-1} \\ = \left(\frac{p}{-\varepsilon_p}\right)^{t_p} c(\frac{-Q_{A_0}(\beta)}{p^{-1}\prod\limits_{\substack{\ell|N\\ \ell\neq p}} \ell^{t_\ell-1} d^2}) \prod_{\substack{\ell|N\\ \ell\neq p}} (-\varepsilon_\ell)^{t_\ell-1}(-\epsilon_p)^{t_p-1} \\ = p^{t_p} c(\frac{-Q_{A_0}(\beta)}{p^{-1}\prod\limits_{\substack{\ell|N\\ \ell\neq p}} \ell^{t_\ell-1} d^2}) \prod_{\substack{\ell|N\\ \ell\neq p}} (-\varepsilon_\ell)^{t_\ell-1}(-\epsilon_p)^{-1}. \end{aligned} \tag{6.12}$$

Hence, as $(-\epsilon_p)^{-1} = -\epsilon_p$, we have

$$\begin{aligned} \sum_{t_\ell=0}^{2u_\ell+\delta_\ell} \sum_{d|n} c(\frac{-Q_{A_0}(\beta)}{\prod\limits_{\ell|N} \ell^{t_\ell-1} d^2}) \prod_{\ell|N} (-\varepsilon_\ell)^{t_\ell-1} \\ = \frac{p^{2u_p+\delta_p+1}-1}{p-1} \sum_{d|n} c(\frac{-Q_{A_0}(\beta)}{p^{-1}\prod\limits_{\substack{\ell|N\\ \ell\neq p}} \ell^{t_\ell-1} d^2}) \prod_{\substack{\ell|N\\ \ell\neq p}} (-\varepsilon_\ell)^{t_\ell-1}(-\epsilon_p). \end{aligned} \tag{6.13}$$

Theorem 6.5. *Let $f \in S(\Gamma_0(N), r)$ be a newform and eigenfunction of the Atkin Lehner involution with eigenvalue ϵ_p at each $p|N$. Let F_f be the lift of f defined in (6.1). Then F_f is a Hecke eigenform with*

$$C_1^{(1)} \cdot F_f = (p^3 + p^2 + p - 1) F_f.$$

Proof. We shall prove the Hecke eigenvalue for the most general case of β with $u_p \geq 1$. The proof for the cases $u_p = 0$ with $\delta_p \in \{0, 1\}$ is similar and follows immediately after substituting for u_p and δ_p. Using (6.13) and $Q_{A_0}(a\beta) = a^2 Q_{A_0}(\beta)$, we have

$$p^2 A(p\beta) = p^3 \sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell | N \\ \ell \neq p}} \sum_{t_\ell = 0}^{2u_\ell + \delta_\ell} \frac{p^{2u_p + \delta_p + 3} - 1}{p - 1} \times \sum_{d | n} c\Big(\frac{-p^2 Q_{A_0}(\beta)}{p^{-1} \prod_{\substack{\ell | N \\ \ell \neq p}} \ell^{t_\ell - 1} d^2}\Big) \prod_{\substack{\ell | N \\ \ell \neq p}} (-\varepsilon_\ell)^{t_\ell - 1} (-\epsilon_p);$$

$$A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell | N \\ \ell \neq p}} \sum_{t_\ell = 0}^{2u_\ell + \delta_\ell} \frac{p^{2u_p + \delta_p + 1} - 1}{p - 1} \times \sum_{d | n} c\Big(\frac{-Q_{A_0}(\beta)}{p^{-1} \prod_{\substack{\ell | N \\ \ell \neq p}} \ell^{t_\ell - 1} d^2}\Big) \prod_{\substack{\ell | N \\ \ell \neq p}} (-\varepsilon_\ell)^{t_\ell - 1} (-\epsilon_p);$$

$$p^2 A(p^{-1}\beta) = p \sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell | N \\ \ell \neq p}} \sum_{t_\ell = 0}^{2u_\ell + \delta_\ell} \frac{p^{2u_p + \delta_p - 1} - 1}{p - 1} \times \sum_{d | n} c\Big(\frac{-p^{-2} Q_{A_0}(\beta)}{p^{-1} \prod_{\substack{\ell | N \\ \ell \neq p}} \ell^{t_\ell - 1} d^2}\Big) \prod_{\substack{\ell | N \\ \ell \neq p}} (-\varepsilon_\ell)^{t_\ell - 1} (-\epsilon_p).$$

Note, $A(p^{-1}\beta) = 0$ if $u_p = 0$. By (6.12) and the fact that $(-\epsilon_p)^2 = 1$, we have

$$\begin{aligned}
&p^2 A(p\beta) + (p^2 - 1) A(\beta) + p^2 A(p^{-1}\beta) \\
&= p \sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell | N \\ \ell \neq p}} \sum_{t_\ell = 0}^{2u_\ell + \delta_\ell} \frac{p^{2u_p + \delta_p + 3} - 1}{p - 1} \sum_{d | n} c\Big(\frac{-Q_{A_0}(\beta)}{p^{-1} \prod_{\substack{\ell | N \\ \ell \neq p}} \ell^{t_\ell - 1} d^2}\Big) \prod_{\substack{\ell | N \\ \ell \neq p}} (-\varepsilon_\ell)^{t_\ell - 1} (-\epsilon_p) \\
&+ (p^2 - 1) \sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell | N \\ \ell \neq p}} \sum_{t_\ell = 0}^{2u_\ell + \delta_\ell} \frac{p^{2u_p + \delta_p + 1} - 1}{p - 1} \\
&\qquad \times \sum_{d | n} c\Big(\frac{-Q_{A_0}(\beta)}{p^{-1} \prod_{\substack{\ell | N \\ \ell \neq p}} \ell^{t_\ell - 1} d^2}\Big) \times \prod_{\substack{\ell | N \\ \ell \neq p}} (-\varepsilon_\ell)^{t_\ell - 1} (-\epsilon_p) \\
&+ p^3 \sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell | N \\ \ell \neq p}} \sum_{t_\ell = 0}^{2u_\ell + \delta_\ell} \frac{p^{2u_p + \delta_p - 1} - 1}{p - 1} \sum_{d | n} c\Big(\frac{Q_{A_0}(\beta)}{p^{-1} \prod_{\substack{\ell | N \\ \ell \neq p}} \ell^{t_\ell - 1} d^2}\Big) \prod_{\substack{\ell | N \\ \ell \neq p}} (-\varepsilon_\ell)^{t_\ell - 1} (-\epsilon_p).
\end{aligned}$$

Hence,

$$\begin{aligned}
&p^2A(p\beta) + (p^2-1)A(\beta) + p^2A(p^{-1}\beta)\\
&= \sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell|N\\ \ell\neq p}} \sum_{t_\ell=0}^{2u_\ell+\delta_\ell} \Big(\frac{(p(p^{2u_p+\delta_p+3}-1)}{p-1} + \frac{(p^2-1)(p^{2u_p+\delta_p+1}-1)}{p-1}\\
&\qquad + \frac{p^3(p^{2u_p+\delta_p-1}-1))}{p-1}\Big) \sum_{d|n} c\Big(\frac{-Q_{A_0}(\beta)}{p^{-1}\prod\limits_{\substack{\ell|N\\ \ell\neq p}} \ell^{t_\ell-1}d^2}\Big) \prod_{\substack{\ell|N\\ \ell\neq p}} (-\varepsilon_\ell)^{t_\ell-1}(-\epsilon_p)\\
&= (p^3+p^2+p-1)\sqrt{Q_{A_0}(\beta)} \sum_{\substack{\ell|N\\ \ell\neq p}} \sum_{t_\ell=0}^{2u_\ell+\delta_\ell} \frac{p^{2u_p+\delta_p+1}-1}{p-1}\\
&\qquad \times \sum_{d|n} c\Big(\frac{-Q_{A_0}(\beta)}{p^{-1}\prod\limits_{\substack{\ell|N\\ \ell\neq p}} \ell^{t_\ell-1}d^2}\Big) \prod_{\substack{\ell|N\\ \ell\neq p}} (-\varepsilon_\ell)^{t_\ell-1}(-\epsilon_p)\\
&= (p^3+p^2+p-1)A(\beta).
\end{aligned}$$

The result now follows from Proposition 6.4 and equation (6.10). □

7. Non-vanishing of the lift

In this section, we will obtain the non-vanishing of the map $f \to F_f$ constructed in Section 4. Let us start by observing that the proof of Lemma 4.5 of [23] can be used to conclude that there exists $M > 0$ such that the Fourier coefficient $c(-M)$ of f is non-zero. If f is a Hecke eigenform, then this implies that $c(-1) \neq 0$. Using the explicit formula (4.5) for the Fourier coefficients for F_f, we can see that in this case we get $A(1) \neq 0$. Hence, the map $f \to F_f$ is injective when restricted to Hecke eigenforms f. We will now prove the injectivity for all f.

Consider a basis of Hecke eigenforms $\{f_1, \cdots, f_k\}$ of $S(\Gamma_0(N), r)$. Since this is a finite set, we can find a prime $p \nmid N$ such that the Hecke eigenvalues $\lambda_p^{(i)}$ of f_i for $i = 1, \cdots k$ satisfy $|\lambda_p^{(i)}| \neq |\lambda_p^{(j)}|$ for all $i \neq j$. This follows from Corollary 4.1.3 of [32]. Let $F_1, \cdots, F_k$ be the lifts of $f_1, \cdots, f_k$. By Theorem 6.2, we know that F_i are Hecke eigenforms with eigenvalues $\mu_{p,1,i} = p^2\Big((\lambda_p^{(i)})^2 + p + p^{-1}\Big)$. Because of the choice of p, we again see that $\mu_{p,1,i} \neq \mu_{p,1,j}$ for all $i \neq j$.

Theorem 7.1. *The map $f \to F_f$ is an injective linear map on $S(\Gamma_0(N), r)$.*

Proof. Let notations be as above the statement of the theorem. Suppose there exist complex numbers $c_1, \cdots, c_k$ such that $c_1F_1 + \cdots + c_kF_k = 0$. Applying the

Hecke operator $C_{3,p}^{(1)}$ $k-1$ times, we get

$$\begin{aligned} c_1F_1 + c_2F_2 + \cdots + c_kF_k &= 0 \\ \mu_{p,1,1}c_1F_1 + \mu_{p,1,2}c_2F_2 + \cdots + \mu_{p,1,k}c_kF_k &= 0 \\ \mu_{p,1,1}^2c_1F_1 + \mu_{p,1,2}^2c_2F_2 + \cdots + \mu_{p,1,k}^2c_kF_k &= 0 \\ \cdots &= \cdots \\ \mu_{p,1,1}^{k-1}c_1F_1 + \mu_{p,1,2}^{k-1}c_2F_2 + \cdots + \mu_{p,1,k}^{k-1}c_kF_k &= 0. \end{aligned}$$

This can be rewritten as

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_{p,1,1} & \mu_{p,1,2} & \cdots & \mu_{p,1,k} \\ \mu_{p,1,1}^2 & \mu_{p,1,2}^2 & \cdots & \mu_{p,1,k}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{p,1,1}^{k-1} & \mu_{p,1,2}^{k-1} & \cdots & \mu_{p,1,k}^{k-1} \end{bmatrix} \begin{bmatrix} c_1F_1 \\ c_2F_2 \\ \cdots \\ \cdots \\ c_kF_k \end{bmatrix} = 0.$$

The matrix on the left hand side is a Vandermonde matrix, with determinant

$$\prod_{1\le i<j\le k} (\mu_{p,1,i} - \mu_{p,1,j}) \neq 0,$$

since all the $\mu_{p,1,i}$'s are distinct. Hence the matrix is invertible, which implies that $c_iF_i = 0$ for all i. But all the F_i are non-zero, so all the $c_i = 0$. This completes the proof of the theorem. □

Remark 7.2. Here, without assuming that f is a Hecke eigenform, we cannot get the non-vanishing as in [23] only using the explicit formula (4.5) for the Fourier coefficients of F_f. The reason is that even though we can find an integer $M > 0$ such that $c(-M) \neq 0$, there is no guarantee that, for an arbitrary maximal order $\mathcal{O}$, there exists $\beta \in \mathcal{O}'$ such that $Q_{A_0}(\beta) = M$.

8. CAP representation associated to the lift

Assume that $f \in S(\Gamma_0(N), r)$ is a newform, and let $F_f \in \mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$ be the corresponding lift defined in (6.1). Let π_F be the representation of $\mathcal{G}(\mathbb{A})$ generated by F_f.

8.1. *Local components of the representation*

8.1.1. *The archimedean component*

Let

$$N_\infty := \{n(x) \mid x \in \mathbb{R}^4\}, \qquad A_\infty := \{a_y \mid y \in \mathbb{R}^+\}$$

for $n(x)$ and a_y as defined in Section 4.1. Let $\delta_s : A_\infty \to \mathbb{C}^\times$ be a quasi-character given by $\delta_s(y) = y^s$ for a parameter $s \in \mathbb{C}$. We can trivially extend δ_s to the parabolic subgroup P_∞ with Langlands decomposition $P_\infty = N_\infty A_\infty M_\infty$

for $M_\infty := \left\{ \begin{pmatrix} 1 & & \\ & m & \\ & & 1 \end{pmatrix} \middle| m \in \mathcal{H}(\mathbb{R}) \right\}$. We define the normalized parabolic induction induced from δ_s by $I_{P_\infty}^{G_\infty}(\delta_s)$. Proposition 5.5 of [21] for $N = 4$ gives us

Proposition 8.1. *The archimedean component of* π_F *is isomorphic to* $I_{P_\infty}^{G_\infty}(\delta_{\sqrt{-1}r})$ *as admissible* G_∞ *module, and irreducible. If* r *is real, namely,* f *satisfies the Selberg conjecture on the minimal eigenvalue of the hyperbolic Laplacian,* π_F *is tempered at the archimedean place.*

Using Theorem 3.1 of [24] and Proposition 6.1, we see that π_F is irreducible. Since F_f is a cusp form, we can conclude that π_F is an irreducible, cuspidal representation of $\mathcal{G}(\mathbb{A})$. Hence, we can decompose $\pi_F = \otimes'_v \pi_v$, where π_v is an irreducible, admissible representation of $\mathcal{G}(\mathbb{Q}_v)$. We have obtained the description of π_∞ above. Next we will describe π_p for finite primes p.

8.1.2. *Non-archimedean component:* $p \nmid N$ *case*

Let p be a prime with $p \nmid N$. Let χ_1, χ_2, χ_3 be unramified characters of $\mathbb{Q}_p^\times$. We get a character χ of the split torus of $\mathcal{G}(\mathbb{Q}_p)$ via

$$\mathrm{diag}(a_1, a_2, a_3, a_3^{-1}, a_2^{-1}, a_1^{-1}) \to \chi_1(a_1)\chi_2(a_2)\chi_2(a_3).$$

Extend this to a character of the minimal parabolic subgroup of $\mathcal{G}(\mathbb{Q}_p)$ by setting it to be trivial on the unipotent radical. By unramified principal series representation of $\mathcal{G}(\mathbb{Q}_p)$ we mean the normalized parabolic induction $I(\chi)$ of $\mathcal{G}(\mathbb{Q}_p)$ induced from χ, the character of the minimal parabolic subgroup.

The argument of the proof of [21, Theorem 5.6] works also for our setting. From Theorem 6.2 we thus deduce the following:

Proposition 8.2. *For primes* $p \nmid N$*, the local component* π_p *of* π_F *is the spherical constituent of the unramified principal series representation* $I(\chi)$ *of* $\mathcal{G}(\mathbb{Q}_p)$ *where the character* χ *corresponds to the three unramified characters* χ_1, χ_2, χ_3 *given by*

$$\chi_1(\varpi_p) = \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2} \right)^2, \chi_2(\varpi_p) = p, \chi_3(\varpi_p) = 1.$$

Here, ϖ_p *is an uniformizer in* $\mathbb{Q}_p$*. Hence,* π_p *is non-tempered for every* $p \nmid N$*.*

8.1.3. *Non-archimedean component:* $p|N$ *case*

Let p be a prime with $p|N$. For an unramified character χ of $\mathbb{Q}_p^\times$, we get a character of the torus of $\mathcal{G}(\mathbb{Q}_p)$ via

$$\mathrm{diag}(y, 1, 1, 1, 1, y^{-1}) \to \chi(y).$$

We can extend this to a character of the maximal parabolic subgroup P by setting it to be trivial on the unipotent radical. The modulus character is given by

$$\delta_P(a_y n(x)) = |y|^4.$$

Define the normalized unramified principal series $I(\chi)$ consisting of all smooth functions $f : \mathcal{G}(\mathbb{Q}_p) \to \mathbb{C}$ satisfying

$$f(a_y n(x)g) = |y|^2 \chi(y) f(g) \quad \text{for all } y \in \mathbb{Q}_p^\times, x \in \mathbb{Q}_p^4, g \in \mathcal{G}(\mathbb{Q}_p).$$

If f_1 is an unramified vector in $I(\chi)$, then the Hecke operator $C_1^{(1)}$ acts on f_1 by a constant. To obtain the constant, using Lemma 6.3, we see that

$$\begin{aligned}\left(C_1^{(1)} f_1\right)(1) &= \int_{\mathcal{G}(\mathbb{Q}_p)} \mathrm{char}_{K_1 c_1^{(1)} K_1}(x) f_1(x) dx \\ &= \sum_{x \in \mathfrak{X}_1} f_1(a_p n(x)) + \sum_{x \in \mathfrak{X}_1} f_1(n(x)) + f_1(a_{p^{-1}}) \\ &= p^4 |p|^2 \chi(p) f_1(1) + (p^2 - 1) f_1(1) + |p^{-1}|^2 \chi(p^{-1}) f_1(1) \\ &= \left(p^2 \chi(p) + p^2 - 1 + p^2 \chi(p^{-1})\right) f_1(1). \end{aligned} \tag{8.1}$$

Proposition 8.3. *Let $p|N$. The local representation π_p is the spherical constituent of the unramified principal series $I(\chi)$ with $\chi(\varpi_p) = p$. The representation π_p is non-tempered.*

Proof. F_f is right invariant under the maximal compact K_p. Hence, π_p is the spherical constituent of an unramified principal series. Comparing (8.1) with the Hecke eigenvalue from Theorem 6.5 we get

$$p^3 + p^2 + p - 1 = p^2 \chi(\varpi_p) + p^2 - 1 + p^2 \chi^{-1}(\varpi_p)$$

implying

$$\chi(\varpi_p) = p \text{ or } p^{-1}.$$

In view of the conjugation by the Weyl group we can take χ so that $\chi(\varpi_p) = p$.

Let us show that π_p is non-tempered. We remark that [21, Theorem 5.2] is not applicable to this case since the assumption "$m \geq 2$" does not hold. If π_p is tempered the matrix coefficient $\langle \pi_p(g) v_0, v_0 \rangle$ with a spherical vector v_0 should belong to $L^{2+\epsilon}(\mathcal{G}(\mathbb{Q}_p))$ for any $\epsilon > 0$. However, calculate the integral of $|\langle \pi_p(g) v_0, v_0 \rangle|^{2+\epsilon}$ over the open domain of $\mathcal{G}(\mathbb{Q}_p)$ as follows:

$$\bigsqcup_{m \in \mathbb{Z}} (p^m, 1_4) K_1.$$

This yields a divergent series $\sum_{m \in \mathbb{Z}} p^{-m(2+\epsilon)} |\langle v_0, v_0 \rangle|^{2+\epsilon}$ and hence $\langle \pi_p(g) v_0, v_0 \rangle$ is not $2 + \epsilon$-integrable for any $\epsilon > 0$, as required. □

8.2. *Cuspidal representation generated by F_f and its CAP property*

Following the description of the local components, we can now state the result for the explicit determination of the cuspidal representation generated by F_f.

Theorem 8.4. *Let f be a newform in $S(\Gamma_0(N), r)$ and let π_F be the cuspidal representation generated by F_f. Then,* *i)* π_F *is irreducible and decomposes into the restricted tensor product* $\pi_F = \otimes'_{v\leq\infty}\pi_v$ *of irreducible admissible representations* π_v *of* $\mathcal{G}(\mathbb{Q}_v)$.

ii) *For* $v = p < \infty$, *if* $p \nmid N$ *then* π_p *is the spherical constituent of the unramified principal series representation of* $\mathcal{G}_p$ *with the Satake parameters*

$$\mathrm{diag}\left(\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, p, 1, 1, p^{-1}, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}\right).$$

iii) *For* $v = p < \infty$, *if* $p \mid N$ *then* π_p *is the spherical constituent of the parabolic induction* $I(\chi)$ *of* $\mathcal{G}(\mathbb{Q}_p)$ *defined by*

$$\chi(p) = p.$$

iv) *For every finite prime* p, π_p *is non-tempered. Suppose that the Selberg conjecture holds for* f, *namely* r *is a real number for the Laplace eigenvalue for* f. *Then* π_∞ *is tempered.*

Proof. This follows from Proposition 8.1, Proposition 8.2 and Proposition 8.3. □

We now review the definition of a CAP representation from [23, Definition 6.6].

Definition 8.5. Let G_1 and G_2 be two reductive algebraic groups over a number field F such that $G_{1,v} \simeq G_{2,v}$ for almost all places v, where $G_{i,v} = G_i(F_v)$ $(i = 1, 2)$ is the group of F_v-points of G_i for the local field F_v at v. Let P_2 be a parabolic subgroup of G_2 with Levi decomposition $P_2 = M_2N_2$. An irreducible cuspidal automorphic representation $\pi = \otimes'_v\pi_v$ of $G_1(\mathbb{A})$ is called *cuspidal associated to parabolic* (CAP) P_2, if there exists an irreducible cuspidal automorphic representation σ of M_2 such that $\pi_v \simeq \pi'_v$ for almost all places v, where $\pi' = \otimes'_v\pi'_v$ is an irreducible constituent of $\mathrm{Ind}_{P_2(\mathbb{A})}^{G_2(\mathbb{A})}(\sigma)$.

For our case $G_1 = \mathcal{G} = \mathrm{O}(1, 5)$ and $G_2 = \mathrm{O}(3, 3)$. We have $G_{1,p} = G_{2,p}$ for all $p \nmid N$. Let σ be a cuspidal representation of GL_2 generated by a Maass cusp form f with the trivial central character. Assume that f is a newform. We want to regard the representation $|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ (cf.[23, Section 6.2]) as the representation of $\mathbb{A}^\times \times \mathrm{O}(2, 2)(\mathbb{A})$, which is isomorphic to a Levi subgroup of a maximal parabolic subgroup $P(\mathbb{A})$ of $\mathrm{O}(3, 3)(\mathbb{A})$. Recall that our previous work [23] introduced the parabolic induction from the representation $|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ to discuss the CAP property of our lifting for the case of $d_B = 2$ in the setting of GL_2 over B. In the present setting we consider the parabolic induction from the aforementioned representation of $\mathbb{A}^\times \times \mathrm{O}(2, 2)(\mathbb{A})$ instead and can show that π_F is a CAP representation attached to this parabolic induction.

To see this we start with recalling the following two isomorphisms (cf. Section 2.3)

$$\mathrm{GL}_2 \times \mathrm{GL}_2/\{(z, z) \mid z \in \mathrm{GL}_1\} \simeq \mathrm{GSO}(2, 2), \quad \mathrm{GO}(2, 2) = \mathrm{GSO}(2, 2) \rtimes \langle t\rangle.$$

We now note that the representation $|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ can be regarded as the representation of $\mathrm{GSO}(2,2)(\mathbb{A})$ since the central character of σ is trivial. We construct a representation of $\mathrm{GO}(2,2)(\mathbb{A})$ by considering its induced representation from $\mathrm{GSO}(2,2)(\mathbb{A})$ to $\mathrm{GO}(2,2)(\mathbb{A})$. Furthermore consider the pull-back of the representation of $\mathrm{GO}(2,2)(\mathbb{A})$ to $\mathbb{A}^\times \times \mathrm{O}(2,2)(\mathbb{A})$ via the surjection $\mathbb{A}^\times \times \mathcal{O}(2,2)(\mathbb{A}) \to \mathrm{GO}(2,2)(\mathbb{A})$. We denote the resulting representation simply by σ and introduce the normalized parabolic induction $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{O}(3,3)(\mathbb{A})}\sigma$, where P is the maximal parabolic subgroup with Levi subgroup isomorphic to $\mathrm{GL}(1) \times \mathrm{O}(2,2)$ and the abelian unipotent radical. Then we have the following:

Proposition 8.6. *Let π_F be as above and recall that we have assumed that the Maass cusp form f is a newform. The cuspidal representation π_F is CAP to the parabolic induction $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{O}(3,3)(\mathbb{A})}\sigma$.*

Proof. We first review the accidental isomorphism $(\mathrm{GL}_4 \times \mathrm{GL}_1)/\{(z \cdot 1_4, z^{-2}) \mid z \in \mathrm{GL}_1\} \simeq \mathrm{GSO}(3,3)$ (see Section 2.3). The restriction of this isomorphism to the GL_4-factor gives rise to the isomorphism of the maximal split tori of the GL_4-factor and $\mathrm{SO}(3,3)$ induced by

$$\mathrm{diag}(x_1, x_2, x_3, x_4) \mapsto \mathrm{diag}(x_1x_2, x_1x_4, x_1x_3, x_2x_4, x_2x_3, x_3x_4)$$

for $x_i \in \mathrm{GL}_1$, $1 \le i \le 4$, where note that $\mathrm{SO}(3,3) = \{(g,z) \in \mathrm{GSO}(3,3) \mid \det(g)z^2 = 1\}$ (cf. [5, Section 3]). In [23, Section 6.1 (6.6), Theorem 6.7], for the GL_4-setting, we have $\mathrm{diag}(a_1, a_2, a_3, a_4)$ with

$$a_1 = p^{1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2},\ a_2 = p^{1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2},\ a_3 = p^{-1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2},$$
$$a_4 = p^{-1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2}$$

as the Satake parameter of the parabolic induction from $|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma$ at a prime $p \nmid N$. Now note that $\mathrm{O}(3,3)$ and $\mathrm{SO}(3,3)$ has the same maximal split torus. In view of the isomorphism of the split tori for PGL_4 and $\mathrm{O}(3,3)$ the corresponding Satake parameter for the $\mathrm{O}(3,3)$-setting is

$$\mathrm{diag}(p, 1, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}, 1, p^{-1}),$$

which is conjugate to the Satake parameter as in Theorem 8.4 under the action of the Weyl group.

We now prove that the parabolic induction $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{O}(3,3)(\mathbb{A})}\sigma$ has the Satake parameter above. We note that by the accidental isomorphism $(\mathrm{GL}_2 \times \mathrm{GL}_2)/\{(z,z) \mid z \in \mathrm{GL}_1\} \simeq \mathrm{GSO}(2,2)$, the Satake parameter $\mathrm{diag}(a_1, a_2, a_3, a_4) = \mathrm{diag}(a_1, a_2) \times$

$\mathrm{diag}(a_3, a_4)$ is mapped to that of $\mathrm{GSO}(2,2)$ given by

$$\mathrm{diag}(\frac{a_1}{a_3}, \frac{a_1}{a_4}, \frac{a_2}{a_3}, \frac{a_2}{a_4}) = \mathrm{diag}(p, p\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, p\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}, p).$$

In addition, we remark that $\mathrm{diag}(p, p, p, p)$ corresponds to the character of the similitude factor of $\mathrm{GSO}(2,2)(\mathbb{Q}_p)$. We thereby see that $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{O}(3,3)(\mathbb{A})}\sigma$ has the desired Satake parameter at $p \nmid N$ since the representation σ, viewed as that of the Levi subgroup of $\mathrm{O}(3,3)(\mathbb{A})$, has the same Satake parameter.

To conclude the proof, as has been pointed out in [21, Section 5.1], we remark that it is valid for the non-connected group $\mathrm{O}(3,3)$ that conjugacy classes of the Satake parameters by the Weyl group classify irreducible unramified principal series, up to isomorphisms. We therefore see that π_F is nearly equivalent to an irreducible constituent of $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{O}(3,3)(\mathbb{A})}\sigma$, as required. □

8.3. *Global standard L-function for F_f*

We define the standard L-function of the orthogonal group $\mathcal{G}$, following Sugano [38, Section 7, (7,6)]. The local factors for places $p \nmid d_B$ are well known. We find them in [38, Section 7, (7,6)]. For places $p|d_B$, the case of $(n_0, \partial) = (4, 2)$ in [38, Section 7 (7.6)] is valid. We define the standard L-function by the Euler product over all finite primes. Putting the local datum of Theorem 8.4 (ii) and (iii) together, we have the following:

Proposition 8.7. *Suppose that a Maass cusp form f is a newform in $S(\Gamma_0(N), r)$ and recall that σ denotes the cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f. Let Π be the irreducible constituent of $\mathrm{Ind}_{P_{2,2}(\mathbb{A})}^{\mathrm{GL}_4(\mathbb{A})}(|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma)$ with Satake parameters as in the proof of Proposition 8.6, where $P_{2,2}$ is the parabolic subgroup of GL_4 with Levi part $\mathrm{GL}_2 \times \mathrm{GL}_2$. By $L(F_f, \mathrm{std}, s)$ (respectively $L(\Pi, \wedge, s)$) we denote the standard L-function for the lift F_f (respectively exterior square L-function of Π). We have*

$$L(F_f, \mathrm{std}, s) = L(\Pi, \wedge, s) = L(\mathrm{sym}^2(f), s)\zeta(s-1)\zeta(s)\zeta(s+1),$$

where the Riemann zeta function $\zeta(s)$ is defined by the Euler product over all finite primes.

Proof. We explain only how to get the equality for the local factors for $p|N$ since the local factors at $p \nmid N$ are calculated in a formal manner by using the explicit formula for the Satake parameters of F_f and Π, where see the proof of Proposition 8.6 for the Satake parameter of Π.

According to [38, Section 7 (7.6)] the local factors of $L(F_f, \mathrm{std}, s)$ are written as

$$(1 - \chi(p)p^{-s})^{-1}(1 - \chi(p)^{-1}p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - p^{-s-1})^{-1}.$$

Now note that, for $p|N$, the local component of the cuspidal representation generated by f is a (twisted) Steinberg representation. From [6, p485] we then know that the local symmetric square L-function $L_p(\mathrm{sym}^2(f), s)$ is $(1 - p^{-(s+1)})^{-1}$ for $p|N$. We thereby obtain the local factors of $L(F_f, \mathrm{std}, s)$ at $p|N$.

We are left with the proof of $L(F_f, \mathrm{std}, s) = L(\Pi, \wedge, s)$ at $p|N$. We use the recent result by Y. Jo [17, Theorem 5.7] to see that the local factor of $L(\Pi, \wedge, s)$ at $p|N$ admits a decomposition into the product

$$L_p(\sigma, \wedge, s+1)L_p(\sigma, \wedge, s-1)L_p(\sigma \times \sigma, s)$$

of the local exterior square L-function and the local Rankin-Selberg L-function for σ. We can verify that the local exterior L-functions of σ at finite primes are nothing but the local Riemann zeta function (cf. [16, Proposition 4.1]). From [6, (1.4.3)] we deduce $L_p(\sigma \times \sigma, s) = \zeta_p(s)\zeta_p(s+1)$. As a result we obtain the desired coincidence $L(F_f, \mathrm{std}, s) = L(\Pi, \wedge, s)$. □

Remark 8.8. The above coincidence of the two L-functions is expected in the framework of the Langlands L-functions (for instance see [5, Section 4]). We remark that our example is given for non-generic representations while the case of generic representations is known to be proved by Shahidi's theory [35, Theorem 3.5] (see [5, Lemma 4.1]).

Acknowledgments

We would like to thank Siegfried Boecherer for several fruitful discussions, in particular for sharing the main idea which led to the proof of non-vanishing in Theorem 7.1. We would also like to thank Abhishek Saha for bringing to our attention the paper [32] which was used in the proof of Theorem 7.1.

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The third author was supported by the ISRAEL SCIENCE FOUNDATION grants No. 376/21 and 421/17.

We would like to thank the referee for the suggestions and comments on an earlier version of the paper.

References

[1] A. Badulescu and D. Renard, Unitary dual of GL(n) at archimedean places and global Jacquet-Langlands correspondence, *Compos. Math.* **146** (2010) 1115–1164.

[2] M. Bhargava, On the Conway-Schneeberger Fifteen Theorem, in *Quadratic forms and their applications (Dublin, 1999)*, Contemp. Math., 272, (Amer. Math. Soc., Providence, RI, 2000), pp. 27–37.

[3] R. E. Borcherds, Automorphic forms with singularities on Grassmanians, *Invent. Math.* **132** (1998) 491–562.

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral transforms.* Vol. I, Based, in part, on notes left by Harry Bateman and Complied by the Staff of the Bateman Manuscript Project, (McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954).

[5] W. T. Gan and S. Takeda, Local Langlands conjecture for $GSp(4)$, *Ann. of Math. (2)* **173** (2011) 1841–1882.
[6] S. Gelbert and H. Jacquet H, A relation between automorphic representations of $GL(2)$ and $GL(3)$, *Ann. Sci. Ecole Normale Sup. (4)* **11** (1978) 471–542.
[7] S. Gelbart and J. Rogawski, L-functions and Fourier-Jacobi coefficients for the unitary group $U(3)$, *Invent. Math.* **105** (1991) 445–472.
[8] D. Ginzburg, A construction of CAP representations in classical groups, *Int. Math. Res. Not.* (2003) no. 20 1123–1140.
[9] D. Ginzburg, D. Jiang and D. Soudry, On CAP Representations for even orthogonal groups I: A correspondence of unramified representations, *Chin. Ann. Math.* Ser. B 36 (2015) 485-–522.
[10] D. Ginzburg, S. Rallis and D. Soudry, Construction of CAP representations for symplectic groups using the descent method. in *Automorphic representations, L-functions and applications: progress and prospects*, Ohio State Univ. Math. Res. Inst. Publ., 11, (de Gruyter, Berlin, 2005), pp. 193–224.
[11] R. Howe and I. I. Piatetskii Shapiro, A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups, in *Automorphic forms, representation theory, and L-functions*, Proc. Sympos. Pure Math., XXXIII part 1, (Amer. Math. Soc., Providence, RI, 1979), pp. 315–322.
[12] T. Ikeda, On the lifting of elliptic cusp forms to Siegel modular forms of degree $2n$, *Ann. of Math. (2)* **154** (2001) 641–681.
[13] T. Ikeda, On the lifting of Hermitian modular forms, *Compositio Math.* **144** (2008) 1107–1154.
[14] T. Ikeda and S. Yamana, On the lifting of Hilbert cusp forms to Hilbert-Siegel cusp forms, *Ann. Sci. Éc. Norm. Supér. (4)* **53** (2020) 1121-–1181.
[15] H. Iwaniec, *Spectral methods of automorphic forms* Second edition. (American Mathematical Society, Providdence RI, Revista Matematica Iberoamericana 2002).
[16] Y. Jo, Factorization of the local exterior square L-functions of GL_m, *Manuscripta math.* **162** (2020) 493–536.
[17] Y. Jo, Derivatives and exceptional poles of the local exterior square L-function for GL_m, *Math. Z.* **294** (2020) 1687–1725.
[18] H. Kim and T. Yamauchi, Cusp forms on the exceptional group of type E_7, *Compositio Math.* **152**, (2016) 223–254.
[19] A. Knapp, *Elliptic Curves.* (MN-40), Volume 40. (Princeton University Press, NJ, 1992).
[20] N. Kurokawa, Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two, *Invent. Math.* **49** (1978) 149–165.
[21] Y. Li, H. Narita and A. Pitale, An explicit construction of non-tempered cusp forms on $O(1, 8n+1)$, *Ann. Math. Québec* **44** (2020) 349–384.
[22] H. Maass Automorphe Funktionen von meheren Veränderlichen und Dirichletsche Reihen, *Abh. Math. Sem. Univ. Hamburg* **16** (1949) 72–100.
[23] M. Muto, H. Narita and A. Pitale, A Lifting to $GL(2)$ over a division quaternion algebra and an explicit construction of CAP representations, *Nagoya Math. J.* **222** (2016) 137–185.
[24] H. Narita, A. Pitale and R. Schmidt, Irreducibility criteria for local and global representations, *Proc. Am. Math. Soc.* **141** (2013) 55–63.
[25] G. Nebe, Even lattices with covering radius $< \sqrt{2}$, *Beiträge Algebra Geom.* **44** (2003) 229–234.
[26] T. Oda, On modular forms associated with indefinite quadratic forms of signature $(2, n-2)$, *Math. Ann.* **231** (1977) 97–144.

[27] I. I. Piatetskii Shapiro, On the Saito-Kurokawa lifting, *Invent. Math.* **71** (1983) 309–338.
[28] A. Pitale, Lifting from $\widetilde{SL}(2)$ to $GSpin(1,4)$, *Int. Math. Res. Notices* (2005) no.63 3919–3966.
[29] S. Rallis, Injectivity properties of lifting associated to Weil representations, *Compositio Math.* **52** (1984) 139–169.
[30] S. Rallis and G. Schiffmann, On a relation between $\widetilde{SL}_2$-cusp forms and cusp forms on tube domains associated to orthogonal groups, *Trans. Amer. Math. Soc.* **263** (1981) 1–58.
[31] S. Rallis and G. Schiffmann, Theta correspondence associated to G_2, *Amer. J. Math.* **111** (1989) 801–849.
[32] D. Ramakrishnan, Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2), *Ann. of Math. (2)* **152** (2000) 45–111.
[33] N. Scheithauer, On the classification of automorphic products and generalized Kac-Moody algebras, *Invent. Math.* **164** (2006) 641–678.
[34] M. Schwagenscheidt and F. Volz, Lifting newforms to vector valued modular forms for the Weil representation, *Int. J. Number Theory* **11** (2015) 2199–2219.
[35] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math. (2)* **132** (1990) 273-330.
[36] G. Shimura, *Arithmetic and analytic theories of quadratic forms and Clifford groups*, Mathematical surveys and monographs, 109, (American Mathematical Society, Providence, RI, 2004).
[37] D. Soudry, The CAP representations of $GSp(4,\mathbb{A})$, *J. Reine Angew. Math.* **383** (1988) 87–108.
[38] T. Sugano, Jacobi forms and the theta lifting, *Commentarii Math. Univ. St. Pauli* **44** (1995) 1-58.
[39] J. Voight, *Quaternion Algebras.* Graduate Texts in Mathematics, 288. (Springer, Cham, 2021).
[40] S. Yamana, On the lifting of elliptic cusp forms to cusp forms on quaternion unitary groups, *J. Number Theory* **130** (2010) 2480–2527.
[41] S. Yamana, On the lifting of Hilbert cusp forms to Hilbert-Hermitian cusp forms, *Trans. Amer. Math. Soc.* **373** (2020) 5395–5438.