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Bessel Models for Lowest Weight Representations of GSp(4, $\mathbb{R}$)

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We prove uniqueness and give precise criteria for existence of split and nonsplit Bessel models for a class of lowest and highest weight representations of the groups GSp(4, $\mathbb{R}$) and Sp(4, $\mathbb{R}$) including all holomorphic and antiholomorphic discrete series representations. Explicit formulas for the resulting Bessel functions are obtained by solving systems of differential equations. The formulas are applied to derive an integral representation for a global L-function on GSp(4) $\times$ GL(2) involving a vector-valued Siegel modular form of degree 2.

1 Introduction

Whittaker models for generic representations of a reductive algebraic group over a local or global field are a very important tool in representation theory. For nongeneric representations of a classical group over an archimedean or non-archimedean local field, Bessel models can sometimes provide a substitute for the missing Whittaker models. Moreover, global Bessel models have been successfully employed to study L-functions and other global objects, as in [2] or [4]. In view of these applications, it is desirable to have as much information as possible about uniqueness and existence of local Bessel models.

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In the present article, we prove uniqueness and give precise criteria for existence of Bessel models for the lowest and highest weight representations of $\mathrm{GSp}(4,\mathbb{R})$ and $\mathrm{Sp}(4,\mathbb{R})$. Our method is elementary and is based on solving a system of linear first-order PDEs. This leads not only to uniqueness and existence criteria, but to explicit formulas for the functions in the Bessel model.

To be a bit more specific, a Bessel model for a representation of $\mathrm{GSp}(4,\mathbb{R})$ consists of a space of functions $B: \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ that transform on the left in a certain way under a character of the *Bessel subgroup*, and such that $\mathrm{GSp}(4,\mathbb{R})$ acts on this space by right translation. The Bessel subgroup $R(\mathbb{R})$ is contained in the Siegel parabolic subgroup, and can be written as a semidirect product $R(\mathbb{R}) = T(\mathbb{R})_0 \ltimes U(\mathbb{R})$, where $U(\mathbb{R})$ is the unipotent radical of the Siegel parabolic subgroup, and where $T(\mathbb{R})_0$ is the identity component of the multiplicative group of a quadratic algebra over $\mathbb{R}$. This quadratic algebra may be $\mathbb{R} \times \mathbb{R}$, in which case we speak of a *split* Bessel model; otherwise we have a *nonsplit* Bessel model.

The class of representations we consider contains not only all the holomorphic and antiholomorphic discrete series representations, but also the limits of discrete series representations and certain nontempered lowest and highest weight modules. This is exactly the class of representations that appear in the automorphic representations generated by holomorphic (scalar or vector valued) Siegel modular forms. In the split case, the existence question has a simple answer: None of the lowest weight representations we consider has a split Bessel model (Corollary 4.2). The nonsplit case is more interesting. In this case $T(\mathbb{R}) \cong \mathbb{C}^\times$, and a Bessel model exists if and only if (an obvious compatibility with the central character is satisfied and) the given character Λ of $T(\mathbb{R})$, restricted to the unit circle, is indexed by an integer m whose absolute value is less than the dimension of the minimal K-type of the lowest weight representation (Theorem 3.10). In particular, the so-called *special* Bessel models, i.e. the models for which Λ is trivial, always exist as long as the central character condition is satisfied.

Bessel models for $\mathrm{GSp}(4)$ have also been treated in the recent preprint [7], both in the real and in the p-adic setting. The overlap with our results consists in the uniqueness and existence criterion for nonsplit Bessel models in the holomorphic discrete series case (see Theorem 3.10 and [7], Theorem 9), which is obtained in [7] by completely different methods. Note that [7] works with Bessel functionals on the smooth vectors of a representation, while we work in the category of $(\mathfrak{g}, K)$-modules. We also mention the works [3] and [9] on *large* discrete series representations. In the former work, a uniqueness result is obtained by using differential equation methods, similar to our approach. In the latter work, which includes the limit of large discrete series case, an existence condition is derived using global methods and theta liftings.

Our focus will be on explicit formulas for Bessel functions, an application of which is given in Section 5. In Theorem 5.1, we will obtain an integral representation for the degree-8 L-function $L(s, \pi_{\mathbf{F}} \times \tau_f)$ of GSp(4) $\times$ GL(2), where $\mathbf{F}$ is a vector-valued holomorphic Siegel modular cusp form of degree 2 (with respect to the full modular group) and f is an elliptic Maaß cusp form (of arbitrary level). The integral representation is based on Furusawa's method [2] and its extension in [6], which involves a global Bessel model for the automorphic representation $\pi_{\mathbf{F}}$ generated by $\mathbf{F}$. The new aspect here is that $\mathbf{F}$ is allowed to be vector-valued, which is possible because of our explicit formulas for the archimedean Bessel functions.

In the first part of this article, we will define the groups and Lie algebras involved and introduce the class of lowest and highest weight representations to be considered. Sections 2.5 and 2.6 will make the notion of Bessel model precise and state some general facts. We start the study of nonsplit Bessel models in Section 3.1 by recalling the important double coset decomposition (20), which already appeared in [2]. In the next section we derive formulas for the action of elements of the complexified Lie algebra on the functions in a Bessel model. These formulas translate into a system of linear first-order PDEs for the lowest weight vector B_0 in a Bessel model. We solve this system whenever possible, leading to our first main result Theorem 3.4 on uniqueness and existence of certain Bessel functions. In Section 3.4, we translate this result into uniqueness and existence statements for Bessel models. This requires some more work, but leads to additional insights into lowest weight modules and their Bessel models.

Split Bessel models will be studied analogously. This time everything is based on the double coset decomposition (87) in Section 4.1. Following that we obtain in Section 4.2 formulas for the action of the complexified Lie algebra on Bessel functions. We remark that these formulas, just as their analogues in the nonsplit case, are independent of the type of representation considered and are potentially useful for the study of representations other than the lowest weight modules considered here. In Section 4.3, we solve the resulting system of differential equations in the split case. It turns out that none of the solutions is of moderate growth, leading immediately to the nonexistence of split Bessel models for lowest weight representations.

Finally, in Sections 5.1–5.3, we demonstrate the applicability of explicit Bessel models by deriving the integral representation for the L-function $L(s, \pi_{\mathbf{F}} \times \tau_f)$ mentioned above. The evaluation of the relevant p-adic zeta integrals has been carried out in [2] and [6], so that we need only evaluate the archimedean integral.

2 Definitions and Preliminaries

2.1 Groups and Lie algebras

Let

$$\mathrm{GSp}(4,\mathbb{R}) = \{g \in \mathrm{GL}(4,\mathbb{R}) : \ {}^t gJg = \mu_2(g)J\}, \qquad J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}.$$

The function $\mu_2 : \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{R}^\times$ is the multiplier homomorphism. Let $\mathrm{Sp}(4,\mathbb{R}) = \{g \in \mathrm{GSp}(4,\mathbb{R}) : \ \mu_2(g) = 1\}$. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{GSp}(4,\mathbb{R})$, and let $\mathfrak{g}^1$ be the Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$. Then $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{g}^1$. Explicitly, $\mathfrak{g}^1 = \{[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in M(4,\mathbb{R}) : \ A = -{}^tD, \ B = {}^tB, \ C = {}^tC\}$. A basis of $\mathfrak{g}^1$ is given by

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad P' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Q' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

A convenient basis for the complexified Lie algebra $\mathfrak{g}^1_{\mathbb{C}} = \mathfrak{g}^1 \otimes \mathbb{C}$ is as follows.

$$Z = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Z' = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$N_+ = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{bmatrix}, \qquad N_- = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{bmatrix},$$

$$X_+ = \frac{1}{2} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad X_- = \frac{1}{2} \begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

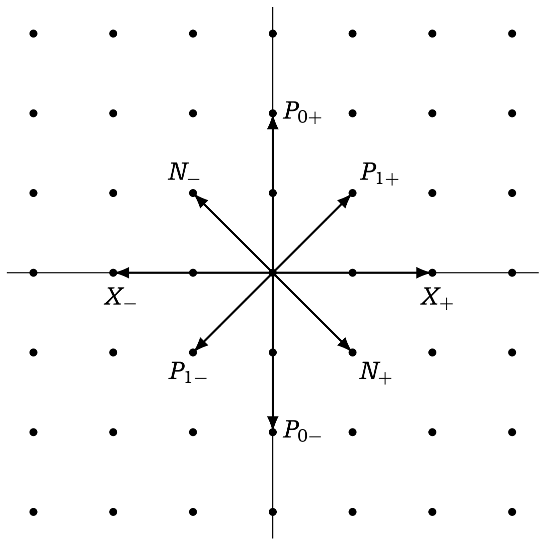
$$P_{1+} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{bmatrix}, \qquad P_{1-} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -1 \\ -i & 0 & -1 & 0 \end{bmatrix},$$

$$P_{0+} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{bmatrix}, \qquad P_{0-} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{bmatrix}.$$

The following multiplication table for this basis will be useful.

	Z	Z'	N_+	N_-	X_+	X_-	P_{1+}	P_{1-}	P_{0+}	P_{0-}
Z	0	0	N_+	$-N_-$	$2X_+$	$-2X_-$	P_{1+}	$-P_{1-}$	0	0
Z'	0	0	$-N_+$	N_-	0	0	P_{1+}	$-P_{1-}$	$2P_{0+}$	$-2P_{0-}$
N_+	$-N_+$	N_+	0	$Z' - Z$	0	$-P_{1-}$	$2X_+$	$-2P_{0-}$	P_{1+}	0
N_-	N_-	$-N_-$	$Z - Z'$	0	$-P_{1+}$	0	$-2P_{0+}$	$2X_-$	0	P_{1-}
X_+	$-2X_+$	0	0	P_{1+}	0	Z	0	N_+	0	0
X_-	$2X_-$	0	P_{1-}	0	$-Z$	0	N_-	0	0	0
P_{1+}	$-P_{1+}$	$-P_{1+}$	$-2X_+$	$2P_{0+}$	0	$-N_-$	0	$Z + Z'$	0	N_+
P_{1-}	P_{1-}	P_{1-}	$2P_{0-}$	$-2X_-$	$-N_+$	0	$-Z - Z'$	0	N_-	0
P_{0+}	0	$-2P_{0+}$	$-P_{1+}$	0	0	0	0	$-N_-$	0	Z'
P_{0-}	0	$2P_{0-}$	0	$-P_{1-}$	0	0	$-N_+$	0	$-Z'$	0

Let $\mathfrak{h}_{\mathbb{R}}$ be the real subspace spanned by Z and Z', and let $\mathfrak{h}_{\mathbb{C}}$ be its complexification. Identifying an $\mathbb{R}$-linear map $\lambda : \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$ with the pair $(\lambda(Z), \lambda(Z'))$, we get an isomorphism $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^2$. Such a map λ is *analytically integral* if $(\lambda(Z), \lambda(Z')) \in \mathbb{Z}^2$. The root system is $\Delta = \{(\pm2, 0), (0, \pm2), (\pm1, \pm1), (\pm1, \mp1)\}$. The following diagram indicates the analytically integral elements, as well as the roots and the elements of the Lie algebra spanning the corresponding root spaces.



Let $\mathfrak{k}$ be the 1-eigenspace of the Cartan involution $\theta : X \mapsto -X^t$, and let $\mathfrak{p}$ the (-1)-eigenspace of θ. Then

$$\mathfrak{k}_{\mathbb{C}} := \mathfrak{k} \otimes \mathbb{C} = \langle Z, Z', N_+, N_- \rangle,$$

and $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}_+ + \mathfrak{p}_-$, with the maximal abelian subalgebras

$$\mathfrak{p}_+ = \langle X_+, P_{1+}, P_{0+} \rangle, \quad \mathfrak{p}_- = \langle X_-, P_{1-}, P_{0-} \rangle.$$

The decomposition $\mathfrak{g}_{\mathbb{C}}^1 = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_-$ holds, and $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_\pm$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}^1$. The compact roots are $\Delta_c = \{(\pm1, \mp1)\}$, and the noncompact roots are $\Delta_n = \{(\pm2, 0), (0, \pm2), (\pm1, \pm1)\}$. The Weyl group W has eight elements and is generated by the reflections about the hyperplanes orthogonal to the root vectors. The compact Weyl group W_K has two elements and is generated by the reflection about the hyperplane orthogonal to the compact roots.

2.2 The maximal compact subgroup

Let K be the standard maximal compact subgroup of $\mathrm{GSp}(4,\mathbb{R})$, and let K^1 be the standard maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{R})$. Then K^1 is the identity component of K, and has index 2 in K. If $\mathfrak{h}_2 = \{Z \in M(2,\mathbb{C}) : Z \text{ symmetric and } \mathrm{Im}(Z) \text{ positive definite}\}$ is the Siegel upper half-space of degree 2, and if $\mathrm{Sp}(4,\mathbb{R})$ acts on $\mathfrak{h}_2$ by fractional linear transformations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$

then K^1 is the stabilizer of the element $I = [^{i}\ _{i}] \in \mathfrak{h}_2$. It is easy to check that $K^1 \simeq \mathrm{U}(2)$ via

$$K^1 \ni \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \longmapsto A + iB \in \mathrm{U}(2).$$

Let $J(h,Z) := CZ + D$. If we let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K^1, then $\mathfrak{k}$ coincides with the Lie algebra of the same name mentioned in the previous section.

2.2.1 Representations of K^1

We say a vector v in a representation of $\mathfrak{k}_{\mathbb{C}}$ has *weight* $(l,l') \in \mathbb{Z}^2$ if $Zv = lv$ and $Z'v = l'v$. To describe the irreducible representations of K, we first describe those of K^1. The latter are in one-one correspondence with the irreducible representations of $\mathfrak{k}_{\mathbb{C}}$ with integral weights. This Lie algebra is a direct sum,

$$\mathfrak{k}_{\mathbb{C}} = \langle Z - Z', N_+, N_-\rangle \oplus \langle Z + Z'\rangle,$$

where $Z + Z'$ spans the center of $\mathfrak{k}_{\mathbb{C}}$. Now the subalgebra $\langle Z - Z', N_+, N_-\rangle$ is isomorphic to $\mathfrak{su}(2)$, and therefore its irreducible representations are indexed by non-negative integers indicating the weight of a highest weight vector (a vector annihilated by N_+). We also can prescribe any integer with which $Z + Z'$ is supposed to act, but have to make sure the resulting representation of $\mathfrak{k}_{\mathbb{C}}$ has integral weights. This shows that the (isomorphism classes of) irreducible representations of K^1 are in one-one correspondence with the set $\{(l,l') \in \mathbb{Z}^2 : l \geq l'\}$, or, in other words, with the analytically integral elements of $\mathfrak{h}_{\mathbb{R}}^*$ modulo the action of W_K. If we let $\rho_{l,l'}$ be the representation corresponding to the pair

(l, l'), then $\rho_{l,l'}$ is characterized by the property that it contains a nonzero vector of weight (l, l') that is annihilated by N_+. The weight structure of $\rho_{l,l'}$ is symmetric with respect to the main diagonal (the wall orthogonal to the compact roots). It contains a highest weight vector of weight (l, l') (annihilated by N_+), and a lowest weight vector of weight (l', l) (annihilated by N_-). It contains the weights "between" these two extremes exactly once. The one-dimensional representations are the $\rho_{l,l}$ with $l \in \mathbb{Z}$. The representation $\rho_{0,0}$ is the trivial one. Evidently, the dimension of $\rho_{l,l'}$ is $l - l' + 1$. The $\rho_{l,l'}$ with trivial central character are those for which l, l' are both even or both odd. These representations are odd-dimensional.

2.2.2 Representations of K

It is now easy to describe the representations of K. They are all obtained by induction from representations of K^1. The induction process has the effect of making the weight structure point symmetric with respect to the origin. More precisely, if (ρ, V) is a representation of K, and if $v \in V$ has weight (l, l'), then $\rho(\mathrm{diag}(1, 1, -1, -1))v$ has weight $(-l, -l')$. Thus the weight structure of an irreducible representation of K combines that of $\rho_{l,l'}$ and $\rho_{-l',-l}$, for some pair (l, l'). The representations $\rho_{l,-l}$ of K^1 with $l \geq 0$ extend in two different ways to representations of K.

2.2.3 Coordinates on K^1

The following coordinates on K^1 will be convenient. We map

$$\mathbb{R}^4 \ni (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \longmapsto r_1(\varphi_1) r_2(\varphi_2) r_3(\varphi_3) r_4(\varphi_4) \in K^1, \tag{1}$$

where

$$r_1(\varphi_1) = \begin{bmatrix} \cos(\varphi_1) & \sin(\varphi_1) & & \\ -\sin(\varphi_1) & \cos(\varphi_1) & & \\ & & \cos(\varphi_1) & \sin(\varphi_1) \\ & & -\sin(\varphi_1) & \cos(\varphi_1) \end{bmatrix}, \quad r_3(\varphi_3) = \begin{bmatrix} \cos(\varphi_3) & & \sin(\varphi_3) & \\ & 1 & & \\ -\sin(\varphi_3) & & \cos(\varphi_3) & \\ & & & 1 \end{bmatrix},$$

$$r_2(\varphi_2) = \begin{bmatrix} \cos(\varphi_2) & & & \sin(\varphi_2) \\ & \cos(\varphi_2) & \sin(\varphi_2) & \\ & -\sin(\varphi_2) & \cos(\varphi_2) & \\ -\sin(\varphi_2) & & & \cos(\varphi_2) \end{bmatrix}, \quad r_4(\varphi_4) = \begin{bmatrix} 1 & & & \\ & \cos(\varphi_4) & & \sin(\varphi_4) \\ & & 1 & \\ & -\sin(\varphi_4) & & \cos(\varphi_4) \end{bmatrix}.$$

One can check that the differential of this map at $(0, 0, 0, 0)$ is regular, so that we get coordinates on K^1 in a neighborhood of the identity. This will suffice: We will only be dealing with K^1-finite functions, which are analytic, and therefore determined on a small neighborhood of the identity of the connected group K^1.

If l, l' are integers, and a function Ψ on K^1 has the property

$$\Psi(gr_3(\varphi_3)r_4(\varphi_4)) = e^{i(l\varphi_3 + l'\varphi_4)}\Psi(g) \qquad \text{for all } \varphi_3, \varphi_4 \in \mathbb{R},$$

then we say that Ψ has *weight* (l, l'). This is equivalent to our previous notion of weight, if Ψ is an element of a function space (e.g. $L^2(K^1)$) on which K^1 acts by right translation.

2.3 Lowest and highest weight representations

We will describe certain lowest and highest weight representations of the group $\mathrm{Sp}(4, \mathbb{R})$. Let l, l' be integers with $l \geq l' > 0$. Then there exists an irreducible representation $\mathcal{E}(l, l')$ of $\mathrm{Sp}(4, \mathbb{R})$ characterized by the existence of a nonzero vector v of weight (l, l') with the property

$$N_+v = X_-v = P_{1-}v = P_{0-}v = 0. \tag{2}$$

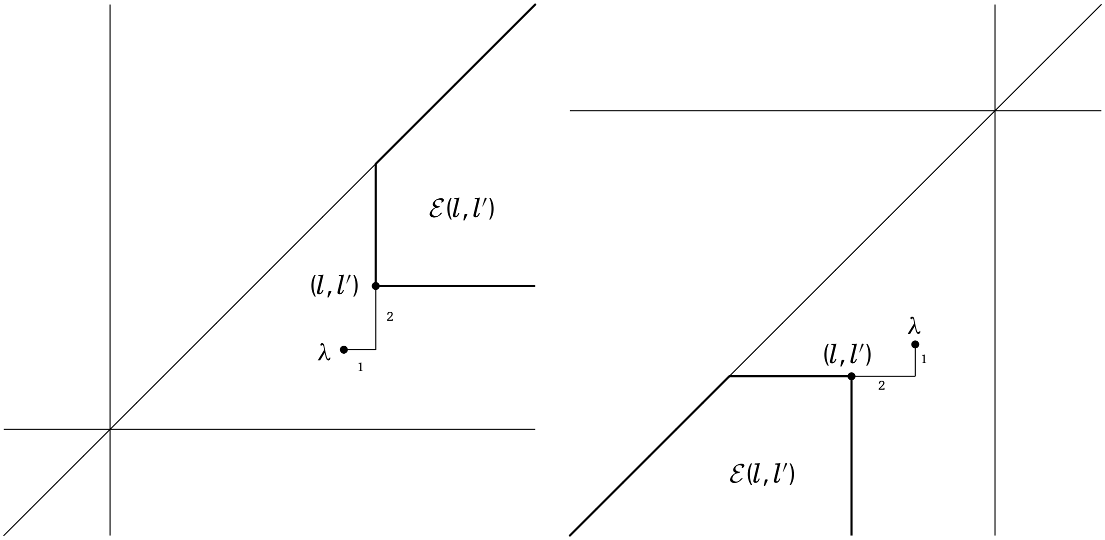
If $l' \geq 3$, then $\mathcal{E}(l, l')$ is a holomorphic discrete series representation with Harish Chandra parameter $\lambda = (l - 1, l' - 2)$. If $l' = 2$, then $\mathcal{E}(l, l')$ is a limit of discrete series representations. If $l' = 1$, then $\mathcal{E}(l, l')$ is not a discrete series or limit of discrete series representation.

Similarly, if $l' \leq l < 0$, then there exists an irreducible representation $\mathcal{E}(l, l')$ characterized by the existence of a nonzero vector v of weight (l, l') with the property

$$N_+v = X_+v = P_{1+}v = P_{0+}v = 0. \tag{3}$$

If $l \leq -3$, then $\mathcal{E}(l, l')$ coincides with the antiholomorphic discrete series representation with Harish Chandra parameter $\lambda = (l + 2, l' + 1)$. If $l = -2$, then $\mathcal{E}(l, l')$ is a limit of discrete series representations. If $l = -1$, then $\mathcal{E}(l, l')$ is not a discrete series or limit of discrete series representation.

The following diagrams illustrate the regions containing all the K^1-types of $\mathcal{E}(l, l')$ in the holomorphic (left) resp. antiholomorphic (right) case.



One reason the lowest weight representations $\mathcal{E}(l,l')$ are important is that they appear as the archimedean components of the automorphic representations generated by Siegel modular forms of degree 2. If $l = l'$, then they correspond to scalar-valued Siegel modular forms of weight l. If $l > l'$, then they correspond to vector-valued Siegel modular forms; see [1]. Using our formulas for Bessel models for $\mathcal{E}(l,l')$ obtained further below, we will demonstrate in Section 5 how to obtain an integral representation for an L-function on $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ involving a vector-valued Siegel modular form.

2.4 Representations of $\mathrm{Sp}(4,\mathbb{R})$ and of $\mathrm{GSp}(4,\mathbb{R})$

Let $\mathrm{Sp}(4,\mathbb{R})^{\pm} = \{g \in \mathrm{GSp}(4,\mathbb{R}) :\ \mu_2(g) = \pm 1\}$. We fix the element $\epsilon = \mathrm{diag}(1,1,-1,-1)$. Then $\mathrm{Sp}(4,\mathbb{R})^{\pm} = \mathrm{Sp}(4,\mathbb{R}) \sqcup \epsilon\mathrm{Sp}(4,\mathbb{R})$. We work in the category of $(\mathfrak{g}, K)$-modules for $\mathrm{GSp}(4,\mathbb{R})$, resp. $(\mathfrak{g}^1, K)$-modules for $\mathrm{Sp}(4,\mathbb{R})^{\pm}$, resp. $(\mathfrak{g}^1, K^1)$-modules for $\mathrm{Sp}(4,\mathbb{R})$. If (π, V) is a $(\mathfrak{g}^1, K^1)$-module of $\mathrm{Sp}(4,\mathbb{R})$, then we define another $(\mathfrak{g}^1, K^1)$-module $(\pi^{\epsilon}, V^{\epsilon})$ by $V^{\epsilon} = V$ and

$$\pi^{\epsilon}(X) = \pi(\mathrm{Ad}(\epsilon)X) \quad (X \in \mathfrak{g}^1), \quad \pi^{\epsilon}(k) = \pi(\epsilon k \epsilon^{-1}) \quad (k \in K^1).$$

If (l,l') is a weight for π, then $(-l,-l')$ is a weight for π^{ϵ}.

2.4.1 Representations of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$

Assume that π is an irreducible $(\mathfrak{g}^1, K^1)$-module. If $\pi^{\varepsilon} \cong \pi$, then π can be extended in exactly two nonisomorphic ways to a $(\mathfrak{g}^1, K)$-module for $\mathrm{Sp}(4,\mathbb{R})^{\pm}$. If $\pi^{\epsilon} \ncong \pi$, then we extend the $(\mathfrak{g}^1, K^1)$-module structure on the direct sum $V \oplus V^\epsilon$ to a $(\mathfrak{g}^1, K)$-module structure by requiring that $\pi(\epsilon)(v_1, v_2) = (v_2, v_1)$. This $(\mathfrak{g}^1, K)$-module is irreducible. We denote it by $I(\pi)$. Every irreducible $(\mathfrak{g}^1, K)$-module is either of the form $I(\pi)$, or is obtained by extending an ϵ-invariant $(\mathfrak{g}^1, K^1)$-module.

Example: Let $\pi = \mathcal{E}(l, l')$ with $l \geq l' > 0$ be a lowest weight representation of $\mathrm{Sp}(4, \mathbb{R})$, as above. We think of $\mathcal{E}(l, l')$ as the underlying $(\mathfrak{g}^1, K^1)$-module. Then π^ϵ is the highest weight representation $\mathcal{E}(-l', -l)$. The induced $(\mathfrak{g}^1, K)$-module combines the space of $\mathcal{E}(l, l')$ and the space of $\mathcal{E}(-l', -l)$. We denote this $(\mathfrak{g}^1, K)$-module by $\mathcal{E}(l, l')$.

2.4.2 Representations of GSp(4, ℝ)

Given a complex number s, we can extend a representation of $\mathrm{Sp}(4, \mathbb{R})^\pm$ to a representation of $\mathrm{GSp}(4, \mathbb{R})^\pm \cong \mathbb{R}_{>0} \times \mathrm{Sp}(4, \mathbb{R})^\pm$ by requiring that $\mathrm{diag}(\gamma, \gamma, \gamma, \gamma)$, $\gamma > 0$, acts by multiplication with γ^s. On the level of Lie algebras, the central element $\mathrm{diag}(1, 1, 1, 1) \in \mathfrak{g}$ acts by multiplication with s. If π is a $(\mathfrak{g}^1, K^1)$-module with $\pi \not\cong \pi^\epsilon$, and $I(\pi)$ is the irreducible $(\mathfrak{g}^1, K)$-module constructed from π, then we denote by $I_s(\pi)$ the extension to a representation of $\mathrm{GSp}(4, \mathbb{R})$. If $\mathcal{E}(l, l')$, $l \geq l' > 0$, is one of the lowest weight $(\mathfrak{g}^1, K)$-modules described in the previous paragraph, then we denote by $\mathcal{E}_s(l, l')$ the corresponding $(\mathfrak{g}, K)$-module. We call these modules lowest weight representations of $\mathrm{GSp}(4, \mathbb{R})$ (even though they have both a lowest and a highest weight, and even though they are not representations of $\mathrm{GSp}(4, \mathbb{R})$ at all).

2.5 Bessel subgroups

Let $U(\mathbb{R}) = \{\left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right] \in \mathrm{GSp}(4, \mathbb{R}) : {}^tX = X\}$. Let S be a nondegenerate real symmetric matrix, and let θ be the character of $U(\mathbb{R})$ given by $\theta(\left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right]) = e^{2\pi i \mathrm{tr}(SX)}$. Explicitly, if $S = \left[\begin{smallmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{smallmatrix}\right]$, then

$$\theta\left(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) = e^{2\pi i(ax+by+cz)}. \tag{4}$$

Let $L = \{\left[\begin{smallmatrix} g & \\ & \lambda\, {}^tg^{-1} \end{smallmatrix}\right] : g \in \mathrm{GL}(2, \mathbb{R}),\ \lambda \in \mathbb{R}^\times\}$ be the Levi component of the Siegel parabolic subgroup of $\mathrm{GSp}(4, \mathbb{R})$. We always think of $\mathrm{GL}(2, \mathbb{R})$ embedded as a subgroup of L via

$g \mapsto [\begin{smallmatrix} g & \\ & \det(g)\,{}^tg^{-1} \end{smallmatrix}]$. Let

$$T(\mathbb{R}) = \{g \in L :\ \theta(u) = \theta(gug^{-1}) \text{ for all } u \in U(\mathbb{R})\} \tag{5}$$

and $T^1(\mathbb{R}) = T(\mathbb{R}) \cap \mathrm{Sp}(4,\mathbb{R})$. Then $T(\mathbb{R}) = \{g \in \mathrm{GL}(2,\mathbb{R}) :\ {}^tgSg = \det(g)S\}$ and $T^1(\mathbb{R}) = \{g \in \mathrm{SL}(2,\mathbb{R}) :\ {}^tgSg = S\}$. Let $T(\mathbb{R})_0$ (resp. $T^1(\mathbb{R})_0$) be the identity component of $T(\mathbb{R})$ (resp. $T^1(\mathbb{R})$) in the real topology. Then $T(\mathbb{R})_0$ contains $\{[\begin{smallmatrix} a & \\ & a \end{smallmatrix}] :\ a > 0\} \cong \mathbb{R}_{>0}$, which, as a subgroup of $\mathrm{GSp}(4,\mathbb{R})$, corresponds to central elements with positive diagonal entries. We have

$$T(\mathbb{R})_0 = \mathbb{R}_{>0} \times T^1(\mathbb{R})_0. \tag{6}$$

We consider two special cases.

(i) Let $S = \pm[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$ (the definite case). Then

$$T(\mathbb{R}) = \left\{\begin{bmatrix} x & y \\ -y & x \end{bmatrix} :\ x, y \in \mathbb{R},\ x^2 + y^2 > 0\right\} \cong \mathbb{C}^\times \tag{7}$$

via $[\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}] \mapsto x + iy$. The subgroup $T^1(\mathbb{R})$ corresponds to elements of the unit circle. In particular, $T(\mathbb{R})$ and $T^1(\mathbb{R})$ are connected.

(ii) Let $S = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$ (the split case). Then

$$T(\mathbb{R}) = \left\{\begin{bmatrix} x & \\ & y \end{bmatrix} :\ x, y \in \mathbb{R}^\times\right\} \cong \mathbb{R}^\times \times \mathbb{R}^\times. \tag{8}$$

This group has four connected components, with $T(\mathbb{R})_0 \cong \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. The group $T^1(\mathbb{R}) \cong \mathbb{R}^\times$ has two connected components, with $T^1(\mathbb{R})_0 \cong \mathbb{R}_{>0}$.

For any S, let

$$R(\mathbb{R}) = T(\mathbb{R})_0 U(\mathbb{R}), \qquad R^1(\mathbb{R}) = T^1(\mathbb{R})_0 U(\mathbb{R}). \tag{9}$$

We call $R(\mathbb{R})$ (resp. $R^1(\mathbb{R})$) the *Bessel subgroup* of $\mathrm{GSp}(4,\mathbb{R})$ (resp. $\mathrm{Sp}(4,\mathbb{R})$) with respect to S. Evidently, $R(\mathbb{R}) = \mathbb{R}_{>0} \times R^1(\mathbb{R})$. Summarizing, we have the following groups and subgroups.

$$
\begin{array}{ccc}
\mathrm{GSp}(4,\mathbb{R}) & = \mathbb{R}_{>0} \times \mathrm{Sp}(4,\mathbb{R})^{\pm} \\
\uparrow & \uparrow \\
\mathrm{GSp}(4,\mathbb{R})^{+} & = \mathbb{R}_{>0} \times \mathrm{Sp}(4,\mathbb{R}) \\
\uparrow & \uparrow \\
R(\mathbb{R}) & = \mathbb{R}_{>0} \times R^{1}(\mathbb{R}) \\
\uparrow & \uparrow \\
T(\mathbb{R})_0 & = \mathbb{R}_{>0} \times T^{1}(\mathbb{R})_0
\end{array}
\tag{10}
$$

2.6 Bessel models

If G is a Lie group, then G acts on the space of smooth functions $\Phi : G \to \mathbb{C}$ by right translation: $(h.\Phi)(g) = \Phi(gh)$. The Lie algebra of G acts on the same space via the derived representation,

$$(X.\Phi)(g) = \frac{d}{dt}\Big|_0 \Phi(g \exp(tX)).$$

We call this action of the Lie algebra also *right translation*.

2.6.1 Growth condition

We define a norm function $\|\cdot\|$ on $\mathrm{GSp}(4,\mathbb{R})$ by

$$\|g\| = \left(\mu_2(g)^{-2} + \sum_{i,j=1}^{4} g_{ij}^2\right)^{1/2}, \qquad g = (g_{ij}) \in \mathrm{GSp}(4,\mathbb{R}), \tag{11}$$

where μ_2 denotes the multiplier. We say a function $\Phi : \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ or $\Phi : \mathrm{Sp}(4,\mathbb{R}) \to \mathbb{C}$ is *slowly increasing* (or of *moderate growth*) if there exist positive constants α, β such that

$$|\Phi(g)| \le \alpha \|g\|^{\beta} \qquad \text{for all } g \in \mathrm{GSp}(4,\mathbb{R}) \quad (\text{resp. } g \in \mathrm{Sp}(4,\mathbb{R})). \tag{12}$$

The norm is designed so that the function $|\mu_2(g)|$ is slowly increasing. A necessary condition for Φ to be slowly increasing is that there exist positive constants α, β such that

$$\left|\Phi\left(\begin{bmatrix}\lambda\zeta&&&\\&\lambda\zeta^{-1}&&\\&&\zeta^{-1}&\\&&&\zeta\end{bmatrix}\right)\right|\le\alpha(\lambda\zeta)^{\beta}\qquad\text{for all }\lambda,\zeta\in\mathbb{R}^{\times},\ |\lambda|,\zeta>1. \tag{13}$$

Evidently, if $\Phi:\ \mathrm{GSp}(4,\mathbb{R})\to\mathbb{C}$ is slowly increasing, then its restriction to $\mathrm{Sp}(4,\mathbb{R})$ is also slowly increasing.

2.6.2 Definition of Bessel models

Let Λ be a character of $T(\mathbb{R})_0$. In the definite case, Λ is a character of $\mathbb{C}^{\times}$, and in the split case, Λ is a character of $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$. Let Λ^1 be the restriction of Λ to $T^1(\mathbb{R})_0$. In view of (6), every character of $T^1(\mathbb{R})_0$ is obtained by such a restriction. Since the elements g of $T(\mathbb{R})_0$ satisfy $\theta(gug^{-1})=\theta(u)$ for all $u\in U(\mathbb{R})$, the map

$$gu\longmapsto\Lambda(g)\theta(u),\qquad g\in T(\mathbb{R})_0,\ u\in U(\mathbb{R}),$$

defines a character of the Bessel subgroup $R(\mathbb{R})$, which we denote by $\Lambda\otimes\theta$. Its restriction to $R^1(\mathbb{R})$ is a character denoted by $\Lambda^1\otimes\theta$.

Let $\mathcal{S}(\Lambda,\theta)$ be the space of functions $B:\ \mathrm{GSp}(4,\mathbb{R})\to\mathbb{C}$ with the following properties.

(i) B is smooth and K-finite.
(ii) $B(tug)=\Lambda(t)\theta(u)B(g)$ for all $t\in T(\mathbb{R})_0$, $u\in U(\mathbb{R})$ and $g\in\mathrm{GSp}(4,\mathbb{R})$.
(iii) B is slowly increasing.

Let $\mathcal{S}^1(\Lambda^1,\theta)$ be the space of functions $B:\ \mathrm{Sp}(4,\mathbb{R})\to\mathbb{C}$ with the following properties.

(i) B is smooth and K-finite.
(ii) $B(tug)=\Lambda^1(t)\theta(u)B(g)$ for all $t\in T^1(\mathbb{R})_0$, $u\in U(\mathbb{R})$ and $g\in\mathrm{Sp}(4,\mathbb{R})$.
(iii) B is slowly increasing.

It is clear that restriction defines a linear map $\mathcal{S}(\Lambda,\theta)\to\mathcal{S}^1(\Lambda^1,\theta)$. We claim that this map is onto. Indeed, let $B\in\mathcal{S}^1(\Lambda^1,\theta)$ be given. First extend B to a function on $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ by setting it equal to zero on elements of negative multiplier. This function satisfies the correct transformation property under elements of $R^1(\mathbb{R})$. Since $\mathrm{GSp}(4,R)=\mathbb{R}_{>0}\times\mathrm{Sp}(4,\mathbb{R})^{\pm}$ and $R(\mathbb{R})=\mathbb{R}_{>0}\times R^1(\mathbb{R})$, we can extend B further to a function on $\mathrm{GSp}(4,\mathbb{R})$ satisfying the correct transformation property under $R(\mathbb{R})$; see (10).

Definition 2.1. Let S be a nondegenerate real symmetric matrix, and let θ be the corresponding character of $U(\mathbb{R})$, as above.

(i) Let (π, V) be a $(\mathfrak{g}, K)$-module. Let Λ be a character of $T(\mathbb{R})_0$. A (Λ, θ)-*Bessel model* for π is a subspace $\mathcal{B}_{\Lambda,\theta}(\pi)$ of $\mathcal{S}(\Lambda, \theta)$, invariant under right translation by $\mathfrak{g}$ and K, such that the $(\mathfrak{g}, K)$-module thus defined is isomorphic to (π, V).

(ii) Let (π, V) be a $(\mathfrak{g}^1, K^1)$-module. Let Λ^1 be a character of $T^1(\mathbb{R})_0$. A (Λ^1, θ)-*Bessel model* for π is a subspace $\mathcal{B}_{\Lambda,\theta}(\pi)$ of $\mathcal{S}^1(\Lambda^1, \theta)$, invariant under right translation by $\mathfrak{g}^1$ and K^1, such that the $(\mathfrak{g}^1, K^1)$-module thus defined is isomorphic to (π, V). □

Our goal in the following is to prove uniqueness and give precise conditions for existence of Bessel models for the lowest and highest weight representations $\mathcal{E}_s(l, l')$ resp. $\mathcal{E}(l, l')$ described above. One necessary condition is obvious: Since the Bessel subgroup $R(\mathbb{R})$ contains the center of $\mathrm{GSp}(4, \mathbb{R})$ (resp. $\mathrm{Sp}(4, \mathbb{R})$), the character Λ, restricted to the center, has to coincide with the central character of the representation.

2.6.3 Change of models

Let S be a nondegenerate real symmetric matrix, as above. Let $A \in \mathrm{GL}(2, \mathbb{R})$ and $\alpha \in \mathbb{R}^\times$ be arbitrary, and define $S' = \alpha\, {}^tASA$. Then we have the two characters

$$\theta\left(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\right) = e^{2\pi i\, \mathrm{tr}(SX)}, \quad \theta'\left(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\right) = e^{2\pi i\, \mathrm{tr}(S'X)}.$$

With $T(\mathbb{R}) = \{g \in \mathrm{GL}(2, \mathbb{R}) : {}^tgSg = \det(g)S\}$ and $T'(\mathbb{R}) = \{g \in \mathrm{GL}(2, \mathbb{R}) : {}^tgS'g = \det(g)S'\}$, there is an isomorphism

$$\begin{aligned} T'(\mathbb{R}) &\xrightarrow{\sim} T(\mathbb{R}), \\ g &\longmapsto AgA^{-1}. \end{aligned}$$

Let Λ be a character of $T(\mathbb{R})_0$, and let Λ' be the character of $T'(\mathbb{R})_0$ corresponding to Λ, i.e. $\Lambda'(g) = \Lambda(AgA^{-1})$ for $g \in T'(\mathbb{R})_0$. Assume that $\mathcal{B}_{\Lambda,\theta}(\pi)$ is a Bessel model for a $(\mathfrak{g}, K)$-module (π, V). Then it is easy to check that, for $B \in \mathcal{B}_{\Lambda,\theta}(\pi)$, the function

$$B'(g) = B\left(\begin{bmatrix} A & \\ & \alpha^{-1}\, {}^tA^{-1} \end{bmatrix} g\right), \quad g \in \mathrm{GSp}(4, \mathbb{R}),$$

satisfies $B'(tug) = \Lambda'(t)\theta'(u)B'(g)$ for $t \in T'(\mathbb{R})_0$ and $u \in U(\mathbb{R})$. Hence, the map $B \mapsto B'$ provides a $(\mathfrak{g}, K)$-isomorphism of $\mathcal{B}_{\Lambda,\theta}(\pi)$ with a (Λ', θ')-Bessel model $\mathcal{B}_{\Lambda',\theta'}(\pi)$. It follows that we need to prove existence and uniqueness of Bessel models only for a class of representatives for quadratic forms S under the operation $S \mapsto \alpha\, {}^t ASA$, $A \in \mathrm{GL}(2, \mathbb{R})$, $\alpha \in \mathbb{R}^\times$. There are only two such classes, represented by $S = \left[\begin{smallmatrix}1&\\&1\end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix}1&\\&-1\end{smallmatrix}\right]$. We call a Bessel model corresponding to $S = \left[\begin{smallmatrix}1&\\&1\end{smallmatrix}\right]$ a *nonsplit Bessel model* and a Bessel model corresponding to $S = \left[\begin{smallmatrix}1&\\&-1\end{smallmatrix}\right]$ a *split Bessel model*.

Similar considerations apply to Bessel models for $\mathrm{Sp}(4, \mathbb{R})$. In this case it is enough to prove existence and uniqueness of Bessel models for a class of representatives for quadratic forms S under the operation $S \mapsto {}^t ASA$, $A \in \mathrm{GL}(2, \mathbb{R})$. There are three such classes, represented by $S = \pm\left[\begin{smallmatrix}1&\\&1\end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix}1&\\&-1\end{smallmatrix}\right]$.

Remark: Let (π, V) be a $(\mathfrak{g}^1, K^1)$-module, and let (π^ϵ, V) be the $(\mathfrak{g}^1, K^1)$-module defined in Section 2.4. Assume that V is a (Λ^1, θ)-Bessel model for π. For $B \in V$, let $\tilde{B}(g) = B(\epsilon g \epsilon^{-1})$, and let $\tilde{V} = \{\tilde{B} : B \in V\}$. Then $\tilde{V}$ is a $(\mathfrak{g}^1, K^1)$-module under right translation, realizing π^ϵ. Evidently, $\tilde{V}$ is a (Λ^1, θ^{-1})-Bessel model for π^ϵ. Hence,

$$\pi \text{ has a } (\Lambda^1, \theta)\text{-Bessel model} \iff \pi^\epsilon \text{ has a } (\Lambda^1, \theta^{-1})\text{-Bessel model}. \qquad (14)$$

2.6.4 Behavior under twisting

Assume that $\mathcal{B}_{\Lambda,\theta}(\pi)$ is a (Λ, θ)-Bessel model for the representation π of $\mathrm{GSp}(4, \mathbb{R})$. Let χ be a character of $\mathbb{R}^\times$. We attach to every $B \in \mathcal{B}_{\Lambda,\theta}(\pi)$ the function $\tilde{B}(g) := \chi(\mu_2(g))B(g)$, where μ_2 is the multiplier homomorphism. Let $\tilde{V}$ be the space of all functions $\tilde{B}$, where B runs through $\mathcal{B}_{\Lambda,\theta}(\pi)$. Then right translation on $\tilde{V}$ defines a representation of $\mathrm{GSp}(4, \mathbb{R})$ isomorphic to the twisted representation $(\chi \otimes \pi)(g) := \chi(\mu_2(g))\pi(g)$. Each $\tilde{B} \in \tilde{V}$ satisfies

$$\tilde{B}(tug) = \chi(\det(t))\Lambda(t)\theta(u)\tilde{B}(g) \qquad \text{for } t \in T(\mathbb{R})_0,\ u \in U(\mathbb{R}),\ g \in \mathrm{GSp}(4, \mathbb{R}).$$

Here, $\det(t)$ is the determinant of t considered as an element of $\mathrm{GL}(2, \mathbb{R})$. Since the multiplier function is slowly increasing, it follows that $\tilde{V}$ provides a $((\chi \circ \det)\Lambda, \theta)$-Bessel model for the twisted representation. Hence,

$$\pi \text{ has a } (\Lambda, \theta)\text{-Bessel model} \iff \chi \otimes \pi \text{ has a } ((\chi \circ \det)\Lambda, \theta)\text{-Bessel model}. \qquad (15)$$

Taking (6) into account, it follows that in proving uniqueness and existence of Bessel models, we may assume, whenever convenient, that the character Λ and the central character of π are trivial on $\mathbb{R}_{>0}$.

2.6.5 Relating Bessel models for Sp(4, ℝ) and GSp(4, ℝ)

GSp(4) to Sp(4): Let (π, V) be a given $(\mathfrak{g}, K)$-module, and assume that $V = \mathcal{B}_{\Lambda,\theta}(\pi)$ is a (Λ, θ)-Bessel model. Assume further that π is irreducible, and that upon restriction to $\mathrm{Sp}(4, \mathbb{R})$ we have $V = V_1 \oplus V_2$ with two nonisomorphic, irreducible $(\mathfrak{g}^1, K^1)$-modules (π_1, V_1) and (π_2, V_2); see Section 2.4. For $i = 1, 2$ let $\tilde{V}_i$ be the space of functions obtained by restricting each function in V_i to $\mathrm{Sp}(4, \mathbb{R})$. The surjective map $V_i \to \tilde{V}_i$ given by restriction is obviously a $(\mathfrak{g}^1, K^1)$-map, and since V_i is irreducible, this map is either zero or an isomorphism. In case it is an isomorphism, the space $\tilde{V}_i$ is a (Λ^1, θ)-Bessel model for π_i. It is clear that not all functions in V can be supported on the nonidentity component of $\mathrm{GSp}(4, \mathbb{R})$, so that at least one of the maps $V_i \to \tilde{V}_i$ must be nonzero. Hence, at least one of π_1 or π_2 admits a (Λ^1, θ)-Bessel model.

Sp(4) to GSp(4): Conversely, let (π, V) be a given $(\mathfrak{g}^1, K^1)$-module for which $\pi \not\cong \pi^\epsilon$, and assume that $V = \mathcal{B}_{\Lambda^1,\theta}(\pi)$ is a (Λ^1, θ)-Bessel model. Let Λ be any extension of Λ^1 to a character of $T(\mathbb{R})_0$; see (6). Given $B \in V$, we extend B to a function on $\mathrm{Sp}(4, \mathbb{R})^\pm$ by setting it equal to zero on elements of negative multiplier, and then further to a function on $\mathrm{GSp}(4, \mathbb{R}) = \mathbb{R}_{>0} \times \mathrm{Sp}(4, \mathbb{R})^\pm$, which has the (Λ, θ)-Bessel transformation property; see the discussion before Definition 2.1. Let $\tilde{V}$ be the space of functions thus obtained, and let $\tilde{V}^\epsilon$ be the space of functions $\mathrm{GSp}(4, \mathbb{R}) \ni g \mapsto B(g\epsilon)$ for $B \in \tilde{V}$. The spaces $\tilde{V}$ and $\tilde{V}^\epsilon$ have zero intersection, since the functions in these spaces are supported on different connected components of $\mathrm{GSp}(4, \mathbb{R})$. The direct sum $\tilde{V} \oplus \tilde{V}^\epsilon$ is a $(\mathfrak{g}, K)$-module under right translation. It is a model for the irreducible $(\mathfrak{g}, K)$-module $I_s(\pi)$ considered in Section 2.4, where s is the complex number defining the extension of Λ^1 to Λ. Clearly, $\tilde{V} \oplus \tilde{V}^\epsilon$ is a (Λ, θ)-Bessel model for $I_s(\pi)$.

We just proved that a (Λ^1, θ)-Bessel model for π leads to a (Λ, θ)-Bessel model for $I_s(\pi)$, and clearly two different models for π would lead to two different models for $I_s(\pi)$. Further below we will prove the uniqueness of Bessel models for $\mathrm{GSp}(4, \mathbb{R})$, which therefore implies uniqueness of Bessel models for $\mathrm{Sp}(4, \mathbb{R})$. It also shows that π and π^ϵ cannot both have a (Λ^1, θ)-Bessel model, since the above construction would lead to two different (Λ, θ)-Bessel models for $I_s(\pi) = I_s(\pi^\epsilon)$. We summarize:

Proposition 2.2. Let Λ be a character of $T(\mathbb{R})_0$, let Λ^1 be its restriction to $T^1(\mathbb{R})_0$, and let $\Lambda|_{\mathbb{R}_{>0}}$ be given by $a \mapsto a^s$ with $s \in \mathbb{C}$; see (6). Let (π, V) be a $(\mathfrak{g}^1, K^1)$-module for which $\pi \not\cong \pi^\epsilon$. Then the following are equivalent.

(i) One of π or π^ϵ has a (Λ^1, θ)-Bessel model.
(ii) Exactly one of π or π^ϵ has a (Λ^1, θ)-Bessel model.
(iii) $I_s(\pi)$ has a (Λ, θ)-Bessel model. □

3 Nonsplit Bessel Models

In this section, we investigate the existence and uniqueness of nonsplit Bessel models for the lowest weight representations of $\mathrm{GSp}(4,\mathbb{R})$ and $\mathrm{Sp}(4,\mathbb{R})$. We shall work with $\mathrm{GSp}(4,\mathbb{R})$ and use the discussion preceding Proposition 2.2 to obtain results for $\mathrm{Sp}(4,\mathbb{R})$. As explained in Section 2.6, we may throughout assume that

$$S=\begin{bmatrix}1&\\&1\end{bmatrix}.$$

3.1 Double coset decomposition

In this section, we will derive representatives for the double coset space $R(\mathbb{R})\backslash\mathrm{GSp}(4,\mathbb{R})/K^1$, where $R(\mathbb{R})=T(\mathbb{R})_0U(\mathbb{R})$. Recall from Section 2.5 that

$$T(\mathbb{R})=\left\{\begin{bmatrix}x&y\\-y&x\end{bmatrix}:\ x,y\in\mathbb{R},\ x^2+y^2>0\right\}\cong\mathbb{C}^\times \tag{16}$$

is a connected group. The subgroup $T^1(\mathbb{R})=T(\mathbb{R})\cap\mathrm{SL}(2,\mathbb{R})$ corresponds to the unit circle, and we have

$$T(\mathbb{R})=\left\{\begin{bmatrix}\gamma&\\&\gamma\end{bmatrix}:\ \gamma>0\right\}\cdot T^1(\mathbb{R}). \tag{17}$$

By the Cartan decomposition,

$$\mathrm{GL}(2,\mathbb{R})^+=\mathrm{SO}(2)\cdot\left\{\begin{bmatrix}\zeta_1&\\&\zeta_2\end{bmatrix}:\ \zeta_1\geq\zeta_2>0,\right\}\cdot\mathrm{SO}(2). \tag{18}$$

Therefore,

$$\begin{aligned}\mathrm{GL}(2,\mathbb{R})^+&=T^1(\mathbb{R})\cdot\left\{\begin{bmatrix}\sqrt{\zeta_1\zeta_2}&\\&\sqrt{\zeta_1\zeta_2}\end{bmatrix}\begin{bmatrix}\sqrt{\zeta_1/\zeta_2}&\\&\sqrt{\zeta_2/\zeta_1}\end{bmatrix}:\ \zeta_1\geq\zeta_2>0\right\}\cdot\mathrm{SO}(2)\\&=T(\mathbb{R})\cdot\left\{\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}:\ \zeta\geq1\right\}\cdot\mathrm{SO}(2).\end{aligned} \tag{19}$$

Using this and the Iwasawa decomposition, it is not hard to see that

$$\mathrm{GSp}(4,\mathbb{R}) = R(\mathbb{R}) \cdot \left\{ \begin{bmatrix} \lambda \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix} : \lambda \in \mathbb{R}^\times,\ \zeta \geq 1 \right\} \cdot K^1; \tag{20}$$

see (4.7) of [2]. Here, $R(\mathbb{R}) = T(\mathbb{R})U(\mathbb{R})$ is the Bessel subgroup defined in (9). One can check that all the double cosets in (20) are disjoint. Recall the coordinates (1) in a neighborhood of the identity of K^1. In the following we let

$$h(\lambda, \zeta, \varphi_1, \varphi_2) := \begin{bmatrix} \lambda \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix} r_1(\varphi_1) r_2(\varphi_2) \tag{21}$$

for $\lambda, \zeta \in \mathbb{R}^\times$ and $\varphi_1, \varphi_2 \in \mathbb{R}$.

3.2 Differential operators

In this section, we will derive explicit formulas for the differential operators given by elements of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ on the functions in a nonsplit Bessel model. Assume that $\mathcal{B}_{\Lambda,\theta}(\pi)$ is a Bessel model for the $(\mathfrak{g}, K)$-module (π, V). For any $B \in \mathcal{B}_{\Lambda,\theta}(\pi)$ we define a function $f = f_B$ on $\mathbb{R}^\times \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$ by

$$f(\lambda, \zeta, \varphi_1, \varphi_2) = B(h(\lambda, \zeta, \varphi_1, \varphi_2)). \tag{22}$$

It follows from (20) and the K-finiteness of B that if B has weight (l, l'), then B is determined by f. If L denotes one of the operators $N_\pm, X_\pm, P_{0\pm}, P_{1\pm}$, then $L.B$ will be determined by the associated function $f_{L.B}$.

We first have to compute the action of the noncomplexified Lie algebra $\mathfrak{g}$. If $L \in \mathfrak{g}$, then by definition

$$(L.B)(h(\lambda, \zeta, \varphi_1, \varphi_2)) = \frac{d}{dt}\bigg|_0 B(h(\lambda, \zeta, \varphi_1, \varphi_2) \exp(tL)).$$

Now, at least for small values of t, we can decompose the argument according to (20):

$$h(\lambda,\zeta,\varphi_1,\varphi_2)\exp(tL)=\begin{bmatrix}g(t)&\\&\det(g(t))\,{}^tg(t)^{-1}\end{bmatrix}\begin{bmatrix}1&&x(t)&y(t)\\&1&y(t)&z(t)\\&&1&\\&&&1\end{bmatrix}$$
$$h(\lambda(t),\zeta(t),\varphi_1(t),\varphi_2(t))\,r_3(\varphi_3(t))r_4(\varphi_4(t)). \tag{23}$$

Here, $g(t)\in T(\mathbb{R})$, and $x(t)$ etc. are smooth functions in a neighborhood of 0 satisfying

$$\begin{aligned}&x(0)=y(0)=z(0)=\varphi_3(0)=\varphi_4(0)=0,\\&\lambda(0)=\lambda,\quad \zeta(0)=\zeta,\quad \varphi_1(0)=\varphi_1,\quad \varphi_2(0)=\varphi_2.\end{aligned}$$

According to (17), we can write

$$g(t)=\gamma(t)\begin{bmatrix}\cos(\delta(t))&\sin(\delta(t))\\-\sin(\delta(t))&\cos(\delta(t))\end{bmatrix} \tag{24}$$

with smooth functions $\gamma(t)$ and $\delta(t)$ such that $\gamma(0)=1$ and $\delta(0)=0$. The character Λ of $T(\mathbb{R})$ is of the form

$$\Lambda\left(\gamma\begin{bmatrix}\cos(\delta)&\sin(\delta)\\-\sin(\delta)&\cos(\delta)\end{bmatrix}\right)=\gamma^se^{im\delta},\qquad \gamma>0,\ \delta\in\mathbb{R}, \tag{25}$$

with some $s\in\mathbb{C}$ and $m\in\mathbb{Z}$. It follows that

$$\begin{aligned}&(L.B)(h(\lambda,\zeta,\varphi_1,\varphi_2))\\
&=\frac{d}{dt}\Big|_0\left(\Lambda(g(t))\theta\left(\begin{bmatrix}1&&x(t)&y(t)\\&1&y(t)&z(t)\\&&1&\\&&&1\end{bmatrix}\right)e^{i(l\varphi_3(t)+l'\varphi_4(t))}B(h(\lambda(t),\zeta(t),\varphi_1(t),\varphi_2(t)))\right)\\
&=\frac{d}{dt}\Big|_0\big(\gamma(t)^se^{im\delta(t)}e^{2\pi i(ax(t)+by(t)+cz(t))}e^{i(l\varphi_3(t)+l'\varphi_4(t))}f(\lambda(t),\zeta(t),\varphi_1(t),\varphi_2(t))\big)\\
&=(s\gamma'(0)+i(m\delta'(0)+l\varphi_3'(0)+l'\varphi_4'(0))+2\pi i(ax'(0)+by'(0)+cz'(0)))f(\lambda,\zeta,\varphi_1,\varphi_2)\\
&\quad+\lambda'(0)\frac{\partial f}{\partial\lambda}(\lambda,\zeta,\varphi_1,\varphi_2)+\zeta'(0)\frac{\partial f}{\partial\zeta}(\lambda,\zeta,\varphi_1,\varphi_2)+\varphi_1'(0)\frac{\partial f}{\partial\varphi_1}(\lambda,\zeta,\varphi_1,\varphi_2)+\varphi_2'(0)\frac{\partial f}{\partial\varphi_2}(\lambda,\zeta,\varphi_1,\varphi_2).\end{aligned} \tag{26}$$

Thus, what we need are the derivatives at 0 of the auxiliary functions $\gamma, \delta, \ldots$. To get these, we differentiate the matrix equation (23) and put $t = 0$. This yields 16 linear equations from which the desired derivatives can be determined. The results are as follows.

(i) Let $L = H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then we have

$$\gamma'(0) = -\frac{1}{2}\cos(2\varphi_2), \delta'(0) = \frac{\zeta^2 \sin(2\varphi_1)}{\zeta^4 - 1}, \lambda'(0) = \lambda\cos(2\varphi_2), \zeta'(0) = \frac{1}{2}\zeta\cos(2\varphi_1),$$
$$x'(0) = -\zeta^2\lambda\sin(2\varphi_1)\sin(2\varphi_2), y'(0) = -\lambda\cos(2\varphi_1)\sin(2\varphi_2), z'(0) = \frac{\lambda\sin(2\varphi_1)\sin(2\varphi_2)}{\zeta^2},$$
$$\varphi_1'(0) = \frac{(1+\zeta^4)\sin(2\varphi_1)}{2(1-\zeta^4)}, \varphi_2'(0) = \frac{1}{2}\sin(2\varphi_2), \varphi_3'(0) = 0, \varphi_4'(0) = 0. \tag{27}$$

(ii) Let $L = H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Then we have

$$\gamma'(0) = -\frac{1}{2}\cos(2\varphi_2),\ \delta'(0) = -\frac{\zeta^2 \sin(2\varphi_1)}{\zeta^4 - 1},\ \lambda'(0) = \lambda\cos(2\varphi_2),\ \zeta'(0) = -\frac{1}{2}\zeta\cos(2\varphi_1),$$
$$x'(0) = -\zeta^2\lambda\sin(2\varphi_1)\sin(2\varphi_2),\ y'(0) = -\lambda\cos(2\varphi_1)\sin(2\varphi_2),\ z'(0) = \frac{\lambda\sin(2\varphi_1)\sin(2\varphi_2)}{\zeta^2},$$
$$\varphi_1'(0) = \frac{(1+\zeta^4)\sin(2\varphi_1)}{2(\zeta^4-1)},\ \varphi_2'(0) = \frac{1}{2}\sin(2\varphi_2), \varphi_3'(0) = 0,\ \varphi_4'(0) = 0. \tag{28}$$

(iii) Let $L = F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then we have

$$\gamma'(0) = 0,\ \delta'(0) = \frac{\zeta^2\cos(2\varphi_1)\sin(2\varphi_2)}{2(1-\zeta^4)},\ \lambda'(0) = 0,\ \zeta'(0) = \frac{\zeta}{4}\sin(2\varphi_1)\sin(2\varphi_2),$$
$$x'(0) = \frac{1}{2}\zeta^2\lambda(\cos(2\varphi_1) + \cos(2\varphi_2)),\ y'(0) = -\frac{\lambda}{2}\sin(2\varphi_1),\ z'(0) = \frac{\lambda(\cos(2\varphi_2) - \cos(2\varphi_1))}{2\zeta^2},$$
$$\varphi_1'(0) = \frac{1}{4}\tan(2\varphi_2) + \frac{\zeta^4+1}{4(\zeta^4-1)}\cos(2\varphi_1)\sin(2\varphi_2),\ \varphi_2'(0) = 0,\ \varphi_3'(0) = -\frac{\sin(\varphi_2)^4}{\cos(2\varphi_2)},$$
$$\varphi_4'(0) = \frac{1}{4}\sin(2\varphi_2)\tan(2\varphi_2). \tag{29}$$

(iv) Let $L = G = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{bmatrix}$. Then we have

$$\gamma'(0) = 0,\ \delta'(0) = \frac{\zeta^2\cos(2\varphi_1)\sin(2\varphi_2)}{2(1-\zeta^4)},\ \lambda'(0) = 0,\ \zeta'(0) = \frac{\zeta}{4}\sin(2\varphi_1)\sin(2\varphi_2),$$
$$x'(0) = \frac{1}{2}\zeta^2\lambda(\cos(2\varphi_1)+\cos(2\varphi_2)),\ y'(0) = -\frac{\lambda}{2}\sin(2\varphi_1),\ z'(0) = \frac{\lambda(\cos(2\varphi_2)-\cos(2\varphi_1))}{2\zeta^2},$$
$$\varphi_1'(0) = \frac{1}{4}\tan(2\varphi_2) + \frac{\zeta^4+1}{4(\zeta^4-1)}\cos(2\varphi_1)\sin(2\varphi_2),\ \varphi_2'(0) = 0,\ \varphi_3'(0) = -\frac{\cos(\varphi_2)^4}{\cos(2\varphi_2)},$$
$$\varphi_4'(0) = \frac{1}{4}\sin(2\varphi_2)\tan(2\varphi_2). \tag{30}$$

(v) Let $L = R = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$. Then we have

$$\gamma'(0) = 0,\ \delta'(0) = \frac{\zeta^2\cos(2\varphi_1)\sin(2\varphi_2)}{2(1-\zeta^4)},\lambda'(0)=0,\zeta'(0) = \frac{\zeta}{4}\sin(2\varphi_1)\sin(2\varphi_2),$$
$$x'(0) = \frac{1}{2}\zeta^2\lambda(\cos(2\varphi_2)-\cos(2\varphi_1)), y'(0) = \frac{\lambda}{2}\sin(2\varphi_1),\ z'(0) = \frac{\lambda(\cos(2\varphi_2)+\cos(2\varphi_1))}{2\zeta^2},$$
$$\varphi_1'(0) = -\frac{1}{4}\tan(2\varphi_2) + \frac{\zeta^4+1}{4(\zeta^4-1)}\cos(2\varphi_1)\sin(2\varphi_2),\ \varphi_2'(0) = 0,\ \varphi_3'(0) = \frac{\sin(2\varphi_2)^2}{4\cos(2\varphi_2)},$$
$$\varphi_4'(0) = -\frac{\sin(\varphi_2)^4}{\cos(2\varphi_2)}. \tag{31}$$

(vi) Let $L = R' = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \end{bmatrix}$. Then we have

$$\gamma'(0) = 0,\ \delta'(0) = \frac{\zeta^2\cos(2\varphi_1)\sin(2\varphi_2)}{2(1-\zeta^4)},\ \lambda'(0) = 0,\ \zeta'(0) = \frac{\zeta}{4}\sin(2\varphi_1)\sin(2\varphi_2),$$
$$x'(0) = \frac{1}{2}\zeta^2\lambda(\cos(2\varphi_2)-\cos(2\varphi_1)),\ y'(0) = \frac{\lambda}{2}\sin(2\varphi_1),\ z'(0) = \frac{\lambda(\cos(2\varphi_2)+\cos(2\varphi_1))}{2\zeta^2},$$
$$\varphi_1'(0) = -\frac{1}{4}\tan(2\varphi_2) + \frac{\zeta^4+1}{4(\zeta^4-1)}\cos(2\varphi_1)\sin(2\varphi_2),\ \varphi_2'(0) = 0,\ \varphi_3'(0) = \frac{\sin(2\varphi_2)^2}{4\cos(2\varphi_2)},$$
$$\varphi_4'(0) = -\frac{\cos(\varphi_2)^4}{\cos(2\varphi_2)}. \tag{32}$$

(vii) Let $L = P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then we have

$$\gamma'(0) = 0,\ \delta'(0) = \frac{\zeta^2 \cos(2\varphi_1)\cos(2\varphi_2)}{1-\zeta^4},\ \lambda'(0) = 0,\ \zeta'(0) = \frac{1}{2}\zeta \cos(2\varphi_2)\sin(2\varphi_1)$$
$$x'(0) = -\zeta^2\lambda \sin(2\varphi_2),\ y'(0) = 0,\ z'(0) = -\frac{\lambda \sin(2\varphi_2)}{\zeta^2}$$
$$\varphi_1'(0) = -\frac{1}{2\cos(2\varphi_2)} + \frac{\zeta^4+1}{2(\zeta^4-1)}\cos(2\varphi_1)\cos(2\varphi_2),\ \varphi_2'(0) = 0,\ \varphi_3'(0) = \tan(2\varphi_2)\cos(\varphi_2)^2,$$
$$\varphi_4'(0) = -\tan(2\varphi_2)\sin(\varphi_2)^2. \quad (33)$$

(viii) Let $L = P' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$. Then we have

$$\gamma'(0) = 0,\ \delta'(0) = \frac{\zeta^2 \cos(2\varphi_1)\cos(2\varphi_2)}{1-\zeta^4},\ \lambda'(0) = 0,\ \zeta'(0) = \frac{1}{2}\zeta \cos(2\varphi_2)\sin(2\varphi_1)$$
$$x'(0) = -\zeta^2\lambda \sin(2\varphi_2),\ y'(0) = 0,\ z'(0) = -\frac{\lambda \sin(2\varphi_2)}{\zeta^2}$$
$$\varphi_1'(0) = \frac{1}{2\cos(2\varphi_2)} + \frac{\zeta^4+1}{2(\zeta^4-1)}\cos(2\varphi_1)\cos(2\varphi_2), \varphi_2'(0) = 0,\ \varphi_3'(0) = -\tan(2\varphi_2)\sin(\varphi_2)^2,$$
$$\varphi_4'(0) = \tan(2\varphi_2)\cos(\varphi_2)^2. \quad (34)$$

(ix) Let $L = Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then we have

$$\gamma'(0) = -\frac{\sin(2\varphi_2)}{2},\ \delta'(0) = 0,\ \lambda'(0) = \lambda\sin(2\varphi_2),\ \zeta'(0) = 0$$
$$x'(0) = \zeta^2\lambda\cos(2\varphi_2)\sin(2\varphi_1),\ y'(0) = \lambda\cos(2\varphi_2)\cos(2\varphi_1),\ z'(0) = -\frac{\lambda\cos(2\varphi_2)\sin(2\varphi_1)}{\zeta^2}$$
$$\varphi_1'(0) = 0,\ \varphi_2'(0) = \sin(\varphi_2)^2,\ \varphi_3'(0) = 0,\ \varphi_4'(0) = 0. \quad (35)$$

(x) Let $L = Q' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Then we have

$$\gamma'(0) = -\frac{\sin(2\varphi_2)}{2},\ \delta'(0) = 0,\ \lambda'(0) = \lambda\sin(2\varphi_2),\ \zeta'(0) = 0$$
$$x'(0) = \zeta^2\lambda\cos(2\varphi_2)\sin(2\varphi_1),\ y'(0) = \lambda\cos(2\varphi_2)\cos(2\varphi_1),\ z'(0) = -\frac{\lambda\cos(2\varphi_2)\sin(2\varphi_1)}{\zeta^2}$$
$$\varphi_1'(0) = 0,\ \varphi_2'(0) = -\cos(\varphi_2)^2,\ \varphi_3'(0) = 0,\ \varphi_4'(0) = 0. \tag{36}$$

Using these coefficients, we obtain from (26) the following formulas for the action of the elements of the complexified Lie algebra.

$$Z.B = lB, \tag{37}$$

$$Z'.B = l'B, \tag{38}$$

$$\begin{aligned}(N_\pm.B)(h(\lambda,\zeta,\varphi_1,\varphi_2)) &= \frac{i}{2}\tan(2\varphi_2)(l'-l)\,f(\lambda,\zeta,\varphi_1,\varphi_2) + \frac{1}{2\cos(2\varphi_2)}\frac{\partial f}{\partial\varphi_1}(h(\lambda,\zeta,\varphi_1,\varphi_2))\\ &\mp\frac{i}{2}\frac{\partial f}{\partial\varphi_2}(h(\lambda,\zeta,\varphi_1,\varphi_2)),\end{aligned} \tag{39}$$

$$\begin{aligned}(X_\pm.B)(h(\lambda,\zeta,\varphi_1,\varphi_2)) &= \Bigg(-\frac{s}{4}\cos(2\varphi_2) + \frac{m}{2}\frac{1}{\zeta^2-\zeta^{-2}}(i\sin(2\varphi_1)\pm\cos(2\varphi_1)\sin(2\varphi_2))\\ &\pm\frac{l}{2}\frac{\sin(\varphi_2)^4+\cos(\varphi_2)^4}{\cos(2\varphi_2)}\mp\frac{l'}{4}\frac{\sin(2\varphi_2)^2}{\cos(2\varphi_2)} - \pi i\lambda(\zeta^2-\zeta^{-2})\sin(2\varphi_1)\sin(2\varphi_2)\\ &\mp\pi\lambda(\zeta^2-\zeta^{-2})\cos(2\varphi_1)\mp\pi\lambda(\zeta^2+\zeta^{-2})\cos(2\varphi_2)\Bigg)f(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\frac{1}{2}\cos(2\varphi_2)\lambda\frac{\partial f}{\partial\lambda}(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\frac{1}{4}\big(\cos(2\varphi_1)\pm i\sin(2\varphi_1)\sin(2\varphi_2)\big)\zeta\frac{\partial f}{\partial\zeta}(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\left(\frac{\zeta^2+\zeta^{-2}}{4(\zeta^2-\zeta^{-2})}\big(-\sin(2\varphi_1)\pm i\cos(2\varphi_1)\sin(2\varphi_2)\big)\pm\frac{i}{4}\tan(2\varphi_2)\right)\\ &\times\frac{\partial f}{\partial\varphi_1}(\lambda,\zeta,\varphi_1,\varphi_2)+\frac{1}{4}\sin(2\varphi_2)\frac{\partial f}{\partial\varphi_2}(\lambda,\zeta,\varphi_1,\varphi_2),\end{aligned} \tag{40}$$

$$\begin{aligned}(P_{1\pm}.B)(h(\lambda,\zeta,\varphi_1,\varphi_2)) &= \Bigg(\mp\frac{is}{2}\sin(2\varphi_2) - \frac{im}{\zeta^2-\zeta^{-2}}\cos(2\varphi_1)\cos(2\varphi_2) + \frac{i(l+l')}{2}\sin(2\varphi_2)\\ &-2\pi i\lambda(\zeta^2+\zeta^{-2})\sin(2\varphi_2)\mp2\pi\lambda(\zeta^2-\zeta^{-2})\cos(2\varphi_2)\sin(2\varphi_1)\Bigg)\\ &\times f(\lambda,\zeta,\varphi_1,\varphi_2)\pm i\sin(2\varphi_2)\lambda\frac{\partial f}{\partial\lambda}(\lambda,\zeta,\varphi_1,\varphi_2)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\cos(2\varphi_2)\sin(2\varphi_1)\zeta\frac{\partial f}{\partial\zeta}(\lambda,\zeta,\varphi_1,\varphi_2)\\
&+\frac{\zeta^2+\zeta^{-2}}{2(\zeta^2-\zeta^{-2})}\cos(2\varphi_1)\cos(2\varphi_2)\frac{\partial f}{\partial\varphi_1}(\lambda,\zeta,\varphi_1,\varphi_2)\\
&\mp\frac{i}{2}\cos(2\varphi_2)\frac{\partial f}{\partial\varphi_2}(\lambda,\zeta,\varphi_1,\varphi_2),
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
(P_{0\pm}.B)(h(\lambda,\zeta,\varphi_1,\varphi_2)) = {}&\Big(-\frac{s}{4}\cos(2\varphi_2)-\frac{m}{2}\frac{1}{\zeta^2-\zeta^{-2}}(i\sin(2\varphi_1)\mp\cos(2\varphi_1)\sin(2\varphi_2))\\
&\mp\frac{l}{4}\frac{\sin(2\varphi_2)^2}{\cos(2\varphi_2)}\pm\frac{l'}{2}\frac{\sin(\varphi_2)^4+\cos(\varphi_2)^4}{\cos(2\varphi_2)}-\pi i\lambda(\zeta^2-\zeta^{-2})\sin(2\varphi_1)\sin(2\varphi_2)\\
&\mp\pi\lambda(\zeta^2+\zeta^{-2})\cos(2\varphi_2)\mp\pi\lambda(\zeta^{-2}-\zeta^2)\cos(2\varphi_1)\Big)f(\lambda,\zeta,\varphi_1,\varphi_2)\\
&+\frac{1}{2}\cos(2\varphi_2)\lambda\frac{\partial f}{\partial\lambda}(\lambda,\zeta,\varphi_1,\varphi_2)\\
&+\frac{1}{4}(-\cos(2\varphi_1)\pm i\sin(2\varphi_1)\sin(2\varphi_2))\zeta\frac{\partial f}{\partial\zeta}(\lambda,\zeta,\varphi_1,\varphi_2)\\
&+\Big(\frac{\zeta^2+\zeta^{-2}}{4(\zeta^2-\zeta^{-2})}(\sin(2\varphi_1)\pm i\cos(2\varphi_1)\sin(2\varphi_2))\mp\frac{i}{4}\tan(2\varphi_2)\Big)\\
&\times\frac{\partial f}{\partial\varphi_1}(\lambda,\zeta,\varphi_1,\varphi_2)+\frac{1}{4}\sin(2\varphi_2)\frac{\partial f}{\partial\varphi_2}(\lambda,\zeta,\varphi_1,\varphi_2).
\end{aligned}
\tag{42}
$$

3.3 Existence and uniqueness of Bessel functions

Let Λ be the character of $\mathbb{C}^\times$ defined in (25), i.e.

$$
\Lambda\left(\gamma\begin{bmatrix}\cos(\delta) & \sin(\delta)\\ -\sin(\delta) & \cos(\delta)\end{bmatrix}\right)=\gamma^s e^{im\delta},\qquad \gamma>0,\quad \delta\in\mathbb{R},
\tag{43}
$$

with $s\in\mathbb{C}$ and $m\in\mathbb{Z}$. Let $\pi=\mathcal{E}(l,l')$ be the lowest weight representation of $\mathrm{GSp}(4,\mathbb{R})$ with minimal K-type (l,l'). Assume that B is a highest weight vector in the minimal K-type in a Bessel model for π of type (Λ,θ). Hence, B satisfies the following conditions.

- ($\mathcal{S}1$) B is smooth and K-finite.
- ($\mathcal{S}2$) $B(tug)=\Lambda(t)\theta(u)B(g)$ for all $t\in T(\mathbb{R})$, $u\in U(\mathbb{R})$, $g\in\mathrm{GSp}(4,\mathbb{R})$.
- ($\mathcal{S}3$) B is slowly increasing.
- ($\mathcal{S}4$) $Z.B=lB$ and $Z'.B=l'B$. Equivalently, $B(gr_3(\varphi_3)r_4(\varphi_4))=e^{i(l\varphi_3+l'\varphi_4)}B(g)$ for all $\varphi_3,\varphi_4\in\mathbb{R}$, $g\in\mathrm{GSp}(4,\mathbb{R})$.
- ($\mathcal{S}5$) $N_+.B=X_-.B=P_{1-}.B=P_{0-}.B=0$.

Conditions ($\mathcal{S}1$)–($\mathcal{S}3$) say that B is an element of the space $\mathcal{S}(\Lambda,\theta)$ defined in Section 2.6. Condition ($\mathcal{S}4$) says that B has weight (l,l'). And condition ($\mathcal{S}5$) means that B is a lowest weight vector.

Let $\mathcal{S}(\Lambda,\theta,l,l')$ be the space of all functions $B:\ \mathrm{GSp}(4,\mathbb{R})\to\mathbb{C}$ satisfying conditions ($\mathcal{S}1$)–($\mathcal{S}5$). It follows from ($\mathcal{S}4$) that $B(-g)=(-1)^{l+l'}B(g)$. On the other hand, by ($\mathcal{S}2$) and (43), $B(-g)=(-1)^mB(g)$. Hence, a necessary condition for $\mathcal{S}(\Lambda,\theta,l,l')$ to be nonzero is that $l+l'+m$ is an even integer.

3.3.1 Uniqueness in the neighborhood of identity

As in (22), we let $f(\lambda,\zeta,\varphi_1,\varphi_2)=B(h(\lambda,\zeta,\varphi_1,\varphi_2))$. From formulas (39)–(42) we find that, in a neighborhood of the identity, the conditions in ($\mathcal{S}5$) are equivalent to the following system of linear first-order PDEs.

$$\frac{\partial f}{\partial\lambda}=\left(\frac{l+l'+s}{2\lambda}-2\pi(\zeta^2+\zeta^{-2})\right)f, \tag{44}$$

$$\begin{aligned}\frac{\partial f}{\partial\zeta}=\Bigg(&-4\pi\lambda(\zeta-\zeta^{-3})-\frac{2m\zeta(\cos(2\varphi_1)\sin(2\varphi_2)+i\sin(2\varphi_1))}{(\zeta^4-1)(\cos(2\varphi_1)+i\sin(2\varphi_1)\sin(2\varphi_2))+(\zeta^4+1)\cos(2\varphi_2)}\\&+\frac{l-l'}{\zeta}\frac{(\zeta^4+1)(\cos(2\varphi_1)+i\sin(2\varphi_1)\sin(2\varphi_2))+(\zeta^4-1)\cos(2\varphi_2)}{(\zeta^4-1)(\cos(2\varphi_1)+i\sin(2\varphi_1)\sin(2\varphi_2))+(\zeta^4+1)\cos(2\varphi_2)}\Bigg)f,\end{aligned} \tag{45}$$

$$\frac{\partial f}{\partial\varphi_1}=\left(\frac{2im\zeta^2\cos(2\varphi_2)-(l-l')(\zeta^4-1)(\sin(2\varphi_1)-i\cos(2\varphi_1)\sin(2\varphi_2))}{(\zeta^4-1)(\cos(2\varphi_1)+i\sin(2\varphi_1)\sin(2\varphi_2))+(\zeta^4+1)\cos(2\varphi_2)}\right)f, \tag{46}$$

$$\frac{\partial f}{\partial\varphi_2}=\left(\frac{2m\zeta^2-(l-l')((\zeta^4+1)\sin(2\varphi_2)-i(\zeta^4-1)\sin(2\varphi_1)\cos(2\varphi_2))}{(\zeta^4-1)(\cos(2\varphi_1)+i\sin(2\varphi_1)\sin(2\varphi_2))+(\zeta^4+1)\cos(2\varphi_2)}\right)f. \tag{47}$$

From (44) we conclude

$$f(\lambda,\zeta,\varphi_1,\varphi_2)=\begin{cases}c_1(\zeta,\varphi_1,\varphi_2)\lambda^{\frac{l+l'+s}{2}}e^{-2\pi\lambda(\zeta^2+\zeta^{-2})} & \text{if }\lambda>0,\\ c_2(\zeta,\varphi_1,\varphi_2)(-\lambda)^{\frac{l+l'+s}{2}}e^{-2\pi\lambda(\zeta^2+\zeta^{-2})} & \text{if }\lambda<0\end{cases} \tag{48}$$

for some functions $c_1(\zeta,\varphi_1,\varphi_2)$ and $c_2(\zeta,\varphi_1,\varphi_2)$. Since we are only interested in slowly increasing solutions, we must have $c_2(\zeta,\varphi_1,\varphi_2)=0$; see (13). Substituting the solution for

$\lambda > 0$ into (45), (46), and (47), we obtain the system

$$\frac{\partial c_1}{\partial \zeta} = \Bigg(-\frac{2m\zeta(\cos(2\varphi_1)\sin(2\varphi_2) + i\sin(2\varphi_1))}{(\zeta^4-1)(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^4+1)\cos(2\varphi_2)} + \frac{l-l'}{\zeta}\frac{(\zeta^4+1)(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^4-1)\cos(2\varphi_2)}{(\zeta^4-1)(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^4+1)\cos(2\varphi_2)}\Bigg)c_1, \tag{49}$$

$$\frac{\partial c_1}{\partial \varphi_1} = \left(\frac{2im\zeta^2\cos(2\varphi_2) - (l-l')(\zeta^4-1)(\sin(2\varphi_1) - i\cos(2\varphi_1)\sin(2\varphi_2))}{(\zeta^4-1)(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^4+1)\cos(2\varphi_2)}\right)c_1, \tag{50}$$

$$\frac{\partial c_1}{\partial \varphi_2} = \left(\frac{2m\zeta^2 - (l-l')((\zeta^4+1)\sin(2\varphi_2) - i(\zeta^4-1)\sin(2\varphi_1)\cos(2\varphi_2))}{(\zeta^4-1)(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^4+1)\cos(2\varphi_2)}\right)c_1 \tag{51}$$

for the function c_1.

Lemma 3.1. Let l, l', m be integers such that $l + l' + m$ is even. Then the system (49), (50), (51) has, up to scalars, the unique solution

$$c_1(\zeta,\varphi_1,\varphi_2) = (\zeta(\cos(\varphi_1)\cos(\varphi_2) + i\sin(\varphi_1)\sin(\varphi_2)) + \zeta^{-1}(\cos(\varphi_1)\sin(\varphi_2) + i\sin(\varphi_1)\cos(\varphi_2)))^m \times ((\zeta^2-\zeta^{-2})(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^2+\zeta^{-2})\cos(2\varphi_2))^{\frac{l-l'-m}{2}}. \tag{52}$$

Alternatively,

$$c_1(\zeta,\varphi_1,\varphi_2) = 2^{-m}(\zeta(\cos(\varphi_1)\cos(\varphi_2) + i\sin(\varphi_1)\sin(\varphi_2)) - \zeta^{-1}(\cos(\varphi_1)\sin(\varphi_2) + i\sin(\varphi_1)\cos(\varphi_2)))^{-m} \times ((\zeta^2-\zeta^{-2})(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^2+\zeta^{-2})\cos(2\varphi_2))^{\frac{l-l'+m}{2}}. \tag{53}$$

□

Proof. A direct calculation shows that the two expressions are equal. It is easily verified that (52) satisfies (49), (50), and (51). Since (52) satisfies (49), any other solution $c(\zeta,\varphi_1,\varphi_2)$ of (49) is of the form

$$c(\zeta,\varphi_1,\varphi_2) = c_3(\varphi_1,\varphi_2)(\zeta(\cos(\varphi_1)\cos(\varphi_2) + i\sin(\varphi_1)\sin(\varphi_2)) + \zeta^{-1}(\cos(\varphi_1)\sin(\varphi_2) + i\sin(\varphi_1)\cos(\varphi_2)))^m((\zeta^2-\zeta^{-2})(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^2+\zeta^{-2})\cos(2\varphi_2))^{\frac{l-l'-m}{2}}. \tag{54}$$

with a function $c_3(\varphi_1,\varphi_2)$. Substituting (54) into (50) and simplifying, we obtain $\frac{\partial c_3}{\partial \varphi_1} = 0$. Substituting (54) into (51) and simplifying, we obtain $\frac{\partial c_3}{\partial \varphi_2} = 0$. Hence c_3 is constant. This proves the uniqueness statement. ■

Note that we have shown that if B is a nonzero function in $\mathcal{S}(\Lambda,\theta,l,l')$, then, in a neighborhood of the identity, it is given by the unique function obtained from (48) and Lemma 3.1.

3.3.2 Necessary condition for existence

The next lemma states that the analyticity of B puts restrictions on the possible characters Λ.

Lemma 3.2. Let Λ be a character of $T(\mathbb{R})$ defined in (25), with $l+l'+m$ even. A necessary condition for $\mathcal{S}(\Lambda,\theta,l,l')$ to be nonzero is $|m| \le l-l'$. □

Proof. Assume there exists a nonzero element $B \in \mathcal{S}(\Lambda,\theta,l,l')$. Being K-finite, B is analytic. As we saw above, in a neighborhood of the identity B is, up to a scalar, given by

$$B(h(\lambda,\zeta,\varphi_1,\varphi_2)) = \begin{cases} c_1(\zeta,\varphi_1,\varphi_2)\lambda^{\frac{l+l'+s}{2}} e^{-2\pi\lambda(\zeta^2+\zeta^{-2})} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0 \end{cases} \tag{55}$$

with c_1 as in (52) or (53). It follows from (55) that there exists an analytic function C_1 on $\mathbb{R}_{>0} \times (-\pi,\pi) \times (-\pi,\pi)$ given by $(\zeta,\varphi_1,\varphi_2) \mapsto c_1(\zeta,\varphi_1,\varphi_2)$ on the domain of the function $c_1(\zeta,\varphi_1,\varphi_2)$. Now, first suppose that $m > l-l'$. Then, from (52), we see that $(1,0,\pi/4)$ is a limit point of the domain of $c_1(\zeta,\varphi_1,\varphi_2)$, which goes to infinity as the argument approaches the limit point. But that contradicts the analyticity of C_1. If $m < -(l-l')$, then we can get a contradiction by a similar argument if we use (53). This completes the proof of the lemma. ■

3.3.3 Extending functions from the neighborhood of identity to the whole group

In the next lemma, we obtain a function on the whole group $\mathrm{GSp}(4,\mathbb{R})$ that has a specified behavior in a neighborhood of the identity. This function will be used below to construct an element $B \in \mathcal{S}(\Lambda,\theta,l,l')$.

Lemma 3.3. The function $w:\ \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ defined by

$$w(h) = i \det\left(J\left(h,\begin{bmatrix} i & \\ & i \end{bmatrix}\right)\right) \det\left(J\left(h,\begin{bmatrix} -i & \\ & i \end{bmatrix}\right)\right)\left(\mathrm{tr}\left(h\left\langle\begin{bmatrix} -i & \\ & i \end{bmatrix}\right\rangle\right) - \mathrm{tr}\left(h\left\langle\begin{bmatrix} i & \\ & i \end{bmatrix}\right\rangle\right)\right) \tag{56}$$

satisfies the following properties.

(i) w is polynomial in the entries of $h = (h_{ij}) \in \mathrm{GSp}(4,\mathbb{R})$.

(ii) If $h = \gamma tuh(\lambda,\zeta,\varphi_1,\varphi_2)r_3(\varphi_3)r_4(\varphi_4)$ with $\gamma \in \mathbb{R}_{>0}$, $t \in T^1(\mathbb{R})$, $u \in U(\mathbb{R})$, $\lambda \in \mathbb{R}^\times$, $\zeta \in \mathbb{R}_{>0}$ and $\varphi_1,\dots,\varphi_4 \in \mathbb{R}$, then

$$w(h) = \gamma^4 \lambda e^{-2i\varphi_4}((\zeta^2 - \zeta^{-2})(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + (\zeta^2+\zeta^{-2})\cos(2\varphi_2)). \tag{57}$$

(iii) For $\gamma \in \mathbb{R}^\times$, $t \in T^1(\mathbb{R})$, $u \in U(\mathbb{R})$, $\varphi_3,\varphi_4 \in \mathbb{R}$ and $h \in \mathrm{GSp}(4,\mathbb{R})$ we have

$$w(\gamma tuhr_3(\varphi_3)r_4(\varphi_4)) = \gamma^4 e^{-2i\varphi_4} w(h). \tag{58}$$

□

Proof. (i) Note that $\mathrm{tr}(h\langle[\begin{smallmatrix} i & \\ & i\end{smallmatrix}]\rangle)$ and $\mathrm{tr}(h\langle[\begin{smallmatrix} -i & \\ & i\end{smallmatrix}]\rangle)$ are rational functions in the entries of h. In fact, $\mathrm{tr}(h\langle[\begin{smallmatrix} -i & \\ & i\end{smallmatrix}]\rangle)$ has zeros in the denominator and is not everywhere defined. However, all denominators are canceled by the determinant factors, so that we obtain an everywhere defined polynomial function in the entries of h.

(ii) Let $h = \gamma tuh(\lambda,\zeta,\varphi_1,\varphi_2)r_3(\varphi_3)r_4(\varphi_4)$ with $u = [\begin{smallmatrix} 1 & X \\ & 1\end{smallmatrix}]$. Calculations show that

- $\det(J(h,[\begin{smallmatrix} i & \\ & i\end{smallmatrix}])) = \gamma^2 e^{-i\varphi_3} e^{-i\varphi_4}$,
- $\det(J(h,[\begin{smallmatrix} -i & \\ & i\end{smallmatrix}])) = \gamma^2 \cos(2\varphi_2) e^{i\varphi_3} e^{-i\varphi_4}$,
- $\mathrm{tr}(h\langle[\begin{smallmatrix} i & \\ & i\end{smallmatrix}]\rangle) = i\lambda(\zeta^2+\zeta^{-2}) + \mathrm{tr}(X)$,
- $\mathrm{tr}(h\langle[\begin{smallmatrix} -i & \\ & i\end{smallmatrix}]\rangle) = \frac{-i\lambda(\zeta^2-\zeta^{-2})}{\cos(2\varphi_2)}(\cos(2\varphi_1) + i\sin(2\varphi_1)\sin(2\varphi_2)) + \mathrm{tr}(X)$.

This proves formula (57).

(iii) The transformation property (58) is easily verified from (56). Note that the elements $r_3(\varphi_3)$ and $r_4(\varphi_4)$ fix both $[\begin{smallmatrix} i & \\ & i\end{smallmatrix}]$ and $[\begin{smallmatrix} -i & \\ & i\end{smallmatrix}]$, whereas the elements $r_1(\varphi_1)$ and $r_2(\varphi_2)$ only fix $[\begin{smallmatrix} i & \\ & i\end{smallmatrix}]$. ■

We now state the main theorem about the existence and uniqueness of nonsplit Bessel functions.

Theorem 3.4. Let $l \geq l' > 0$ be integers. Let Λ be the character of $T(\mathbb{R})$ defined by a pair $(s,m) \in \mathbb{C}\times\mathbb{Z}$, as in (43). If $l+l'+m$ is odd, then $\mathcal{S}(\Lambda,\theta,l,l') = 0$. If $l+l'+m$ is even, then

$$\dim_{\mathbb{C}}(\mathcal{S}(\Lambda,\theta,l,l')) = \begin{cases} 1 & \text{if } |m| \leq l-l', \\ 0 & \text{if } |m| > l-l'. \end{cases} \tag{59}$$

Assume that $l+l'+m$ is even and $|m| \le l-l'$, and let B_0 be a nonzero element of $\mathcal{S}(\Lambda,\theta,l,l')$. Then, for all $\lambda \in \mathbb{R}^\times$, $\zeta > 0$ and $\varphi_1,\varphi_2 \in \mathbb{R}$, we have, up to a scalar,

$$B_0(h(\lambda,\zeta,\varphi_1,\varphi_2)) = \begin{cases} c_1(\zeta,\varphi_1,\varphi_2)\lambda^{\frac{l+l'+s}{2}} e^{-2\pi\lambda(\zeta^2+\zeta^{-2})} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0, \end{cases} \tag{60}$$

where c_1 is the function given in Lemma 3.1. Here, $h(\lambda,\zeta,\varphi_1,\varphi_2)$ is as in (21). Moreover, there exist analytic functions $A_j : K^1 \to \mathbb{C}$, for $j \in \{-(l-l'),\ldots,l-l'\}$, which are zero for $j \not\equiv l-l' \bmod 2$, such that

$$B_0(h(\lambda,\zeta,0,0)k) = \left(\sum_{j=-(l-l')}^{l-l'} A_j(k)\zeta^j\right)\lambda^{\frac{l+l'+s}{2}} e^{-2\pi\lambda(\zeta^2+\zeta^{-2})} \tag{61}$$

for all $\lambda,\zeta > 0$ and $k \in K^1$. □

Proof. We saw that, up to a scalar, any nonzero element B of $\mathcal{S}(\Lambda,\theta,l,l')$ coincides with the function (60) in a neighborhood of the identity; note that, by (20), the transformation properties of B determine it on elements of the form $h(\lambda,\zeta,\varphi_1,\varphi_2)$. Since analytic functions on the identity component of $\mathrm{GSp}(4,\mathbb{R})$ are determined in a neighborhood of the identity, it follows that $\dim_{\mathbb{C}}(\mathcal{S}(\Lambda,\theta,l,l')) \le 1$. It also follows that (60) holds for *all* $\lambda \in \mathbb{R}^\times$, $\zeta > 0$ and $\varphi_1,\varphi_2 \in \mathbb{R}$. Looking at the formula (52) for c_1 and assuming that $|m| \le l-l'$, we obtain (61) in a neighborhood of the identity, and then by analyticity in general.

In Lemma 3.2, we have already shown that $\mathcal{S}(\Lambda,\theta,l,l') = 0$ if $|m| > l-l'$. Assuming that $|m| \le l-l'$, it remains to find a function satisfying the conditions ($\mathcal{S}$1)–($\mathcal{S}$5) defining the space $\mathcal{S}(\Lambda,\theta,l,l')$. Let $h = (h_{ij}) \in \mathrm{GSp}(4,\mathbb{R})$. Let w be the polynomial function from Lemma 3.3. If $0 \le m \le l-l'$, we define

$$B(h) = \mu_2(h)^{l'+\frac{s}{2}+\frac{m}{2}} \det(J(h,iI_2))^{-l} w(h)^{\frac{l-l'-m}{2}} (h_{44}-h_{32}+ih_{42}+ih_{34})^m e^{2\pi i \mathrm{tr}(h\langle iI_2\rangle)} \tag{62}$$

if $\mu_2(h) > 0$, and $B(h) = 0$ if $\mu_2(h) < 0$. If $-(l-l') \le m \le 0$, we define

$$B(h) = \mu_2(h)^{l'+\frac{s}{2}-\frac{m}{2}} \det(J(h,iI_2))^{-l} w(h)^{\frac{l-l'+m}{2}} (h_{44}-h_{32}+ih_{42}+ih_{34})^{-m} e^{2\pi i \mathrm{tr}(h\langle iI_2\rangle)} \tag{63}$$

if $\mu_2(h) > 0$, and $B(h) = 0$ if $\mu_2(h) < 0$. Then B is a well-defined analytic function on $\mathrm{GSp}(4,\mathbb{R})$, and we shall prove it satisfies the conditions ($\mathcal{S}$1)–($\mathcal{S}$5). Conditions ($\mathcal{S}$2) and

($\mathcal{S}4$) are verified by a direct calculation, observing iii) of Lemma 3.3. Let $h \in \mathrm{GSp}(4,\mathbb{R})$. In a neighborhood of the identity we can write $h = tuh(\lambda,\zeta,\varphi_1,\varphi_2)r_3(\varphi_3)r_4(\varphi_4)$ with $t \in T(\mathbb{R})$, $u \in U(\mathbb{R})$, λ and ζ close to 1 and $\varphi_1,\dots,\varphi_4$ close to 0; see (20). Let $t = [^{g}\ {}_{\det(g)^t g^{-1}}]$ where $g = \gamma[\begin{smallmatrix}\cos(\delta) & \sin(\delta)\\ -\sin(\delta) & \cos(\delta)\end{smallmatrix}]$ and let $u = [\begin{smallmatrix}1 & X\\ 0 & 1\end{smallmatrix}]$. Calculations show that

(i) $\mu_2(h) = \gamma^2\lambda$,

(ii) $\det(J(h, iI_2)) = \gamma^2 e^{-i\varphi_3}e^{-i\varphi_4}$,

(iii) $\mathrm{tr}(h\langle iI_2\rangle) = i\lambda(\zeta^2+\zeta^{-2}) + \mathrm{tr}(X)$,

(iv) $h_{44} - h_{32} + ih_{42} + ih_{34} = \gamma e^{i\delta}e^{-i\varphi_4}(\zeta(\cos(\varphi_1)\cos(\varphi_2) + i\sin(\varphi_1)\sin(\varphi_2))$
$+\zeta^{-1}(\cos(\varphi_1)\sin(\varphi_2) + i\sin(\varphi_1)\cos(\varphi_2)))$,

(v) $h_{44} + h_{32} + ih_{42} - ih_{34} = \gamma e^{-i\delta}e^{-i\varphi_4}(\zeta(\cos(\varphi_1)\cos(\varphi_2) + i\sin(\varphi_1)\sin(\varphi_2))$
$-\zeta^{-1}(\cos(\varphi_1)\sin(\varphi_2) + i\sin(\varphi_1)\cos(\varphi_2)))$.

Substituting the above formulas into (62) (resp. (63)), we conclude that B, as defined in (62) (resp. (63)), is given by the formula (55) in a neighborhood of the identity. It follows that B satisfies ($\mathcal{S}5$) in a neighborhood of the identity, and then, by analyticity, everywhere. The condition $N_+.B = 0$ implies that B is K-finite. The functions $\mu_2(h)$ and $1/\det(J(h, iI_2))$ are slowly increasing. Since $w(h)$ is a polynomial function, it follows from (62) and (63) that $B(h)$ is slowly increasing. This concludes the proof that $B \in \mathcal{S}(\Lambda,\theta,l,l')$. ■

3.4 Nonsplit Bessel models for lowest weight representations

In this section, we will use Theorem 3.4 to obtain the existence and uniqueness of Bessel models for the lowest weight representations of GSp(4, ℝ). Let $\mathcal{U}(\mathfrak{g}^1_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}^1_{\mathbb{C}} = \mathfrak{sp}(4,\mathbb{C})$. If B is any K^1-finite function on GSp(4, ℝ), then $\mathcal{U}(\mathfrak{g}^1_{\mathbb{C}})B$ is a $(\mathfrak{g}^1, K^1)$-module. Similarly, if B is any K-finite function on GSp(4, ℝ), then $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})B$ is a $(\mathfrak{g}, K)$-module. Assume that B_0 is the nonzero element of $\mathcal{S}(\Lambda,\theta,l,l')$ described in Theorem 3.4. Since $\mathfrak{p}_-.B = N_+.B = 0$ and $Z.B$ and $Z'.B$ are multiples of B, it follows that $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})B_0$ is spanned by functions of the form

$$X_+^{\alpha}P_{1+}^{\beta}P_{0+}^{\gamma}N_-^{\delta}B_0, \quad \alpha,\beta,\gamma,\delta \geq 0,\ \delta \leq l-l'. \tag{64}$$

The main ingredient in the proof of the existence and uniqueness of the Bessel models is the irreducibility of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})B_0$. We will first state a few lemmas that will be used in the proof of the irreducibility of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})B_0$.

3.4.1 General shape of elements in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})B_0$

Lemma 3.5. Let Λ be the character of $T(\mathbb{R})$ defined by a pair $(s,m)\in\mathbb{C}\times\mathbb{Z}$, as in (43). Let $l\geq l'>0$ be integers. We assume that $l+l'+m$ is even and that $|m|\leq l-l'$, so that $\mathcal{S}(\Lambda,\theta,l,l')$ is one-dimensional; see Theorem 3.4. Let B_0 be a function spanning this space. Then every function $B\in\mathcal{U}(\mathfrak{g}_{\mathbb{C}})B_0$ is zero on the nonidentity component of $\mathrm{GSp}(4,\mathbb{R})$. On the identity component, the function $B(h(\lambda,\zeta,0,0)k)$, $\lambda,\zeta>0$, $k\in K^1$, is a linear combination of functions of the form

$$\left(\sum_{j=-(l-l')}^{l-l'}A_j(k)\zeta^j\right)(\zeta^2+\zeta^{-2})^e(\zeta^2-\zeta^{-2})^fP(\lambda)\lambda^{e+f+\frac{l+l'+s}{2}}e^{-2\pi\lambda(\zeta^2+\zeta^{-2})}, \tag{65}$$

where each A_j is an analytic function on K^1, zero if $j\not\equiv l-l' \bmod 2$, where $e,f\in\mathbb{Z}$, $e\geq 0$, and where P is a polynomial in λ (the functions A_j and P, and the exponents e,f all depend on B). □

Proof. By (61), at least the vector B_0 has the asserted property (with $e=f=0$). To prove it in general, it is enough to show that if B has the asserted property, and if X is one of the operators $X_\pm$, $P_{1\pm}$, $P_{0\pm}$, or $N_\pm$, then $X.B$ also has the asserted property; see (64). It is clear that if B is zero on the nonidentity component, then $X.B$ is as well. To prove that $(X.B)(h(\lambda,\zeta,0,0)k)$ is a linear combination of functions of the form (74), we investigate each term in the formulas (39)–(42) for the differential operators. Observing that

$$\lambda\frac{\partial}{\partial\lambda}e^{-2\pi\lambda(\zeta^2+\zeta^{-2})}=-2\pi\lambda(\zeta^2+\zeta^{-2})e^{-2\pi\lambda(\zeta^2+\zeta^{-2})},$$

$$\zeta\frac{\partial}{\partial\zeta}((\zeta^2+\zeta^{-2})^e(\zeta^2-\zeta^{-2})^f)=2e(\zeta^2+\zeta^{-2})^{e-1}(\zeta^2-\zeta^{-2})^{f+1}+2f(\zeta^2+\zeta^{-2})^{e+1}(\zeta^2-\zeta^{-2})^{f-1},$$

$$\zeta\frac{\partial}{\partial\zeta}e^{-2\pi\lambda(\zeta^2+\zeta^{-2})}=-4\pi\lambda(\zeta^2-\zeta^{-2})e^{-2\pi\lambda(\zeta^2+\zeta^{-2})},$$

and that the operator $\lambda\frac{\partial}{\partial\lambda}$ (resp. $\zeta\frac{\partial}{\partial\zeta}$) does not change the degree of a λ-polynomial (resp. ζ-polynomial), it follows that each time the exponent of one of $\zeta^2+\zeta^{-2}$ or $\zeta^2-\zeta^{-2}$ is increased, the other is decreased or the exponent of λ is increased. This completes the proof. ■

Remark. The proof shows slightly more, namely that if B is the function in (64), then $B(h(\lambda,\zeta,0,0)k)$ is a linear combination of functions of the form (65), where $e+f+\deg(P)\leq\alpha+\beta+\gamma$. □

3.4.2 Convergence of certain integrals

Lemma 3.6. Let $\alpha, \delta \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{Z}$. Then the integral

$$\int_0^\infty \int_1^\infty \zeta^\alpha (\zeta^2 - \zeta^{-2})^\beta (\zeta^2 + \zeta^{-2})^\gamma \lambda^\delta e^{-4\pi\lambda(\zeta^2+\zeta^{-2})} \, d\zeta \, d\lambda \tag{66}$$

is convergent if and only if $\delta > -1$ and $\alpha + 2(\beta + \gamma) < 2\delta + 1$. □

Proof. Since the function is positive, we may consider iterated integrals in any order. Carrying out the λ-integration first, we see that this integral is divergent if $\delta \leq -1$. Assume that $\delta > -1$. Then

$$\int_0^\infty \lambda^\delta e^{-4\pi\lambda(\zeta^2+\zeta^{-2})} \, d\lambda = (4\pi)^{-(\delta+1)} \Gamma(\delta + 1)(\zeta^2 + \zeta^{-2})^{-(\delta+1)}.$$

Hence (66) is convergent if and only if

$$\int_1^\infty \zeta^\alpha (\zeta^2 - \zeta^{-2})^\beta (\zeta^2 + \zeta^{-2})^{\gamma-\delta-1} \, d\zeta < \infty. \tag{67}$$

Note that $\int_1^2 (\zeta^2 - \zeta^{-2})^\beta \, d\zeta$ is finite for *all* integers β, so that the behavior at ∞ determines the convergence. The integral (67) therefore converges if and only if $\int_1^\infty \zeta^{\alpha+2(\beta+\gamma-\delta-1)} \, d\zeta$ converges. This is the case if and only if the exponent is less than -1. The assertion follows. ■

Next we consider scalar products of Bessel functions. Note that if Λ is a unitary character, and B_1, B_2 have the (Λ, θ) Bessel transformation property, then the function $B_1(g)\overline{B_2(g)}$ is left $R(\mathbb{R})$ invariant. For any measurable function on $R(\mathbb{R})\backslash G(\mathbb{R})$ we have the integration formula

$$\int_{R(\mathbb{R})\backslash \mathrm{GSp}(4,\mathbb{R})} f(g) \, dg = \int_{K^1} \int_{\mathbb{R}^\times} \int_1^\infty f\left(\begin{bmatrix} \lambda\zeta & & & \\ & \lambda\zeta^{-1} & & \\ & & \zeta^{-1} & \\ & & & \zeta \end{bmatrix} k \right) \frac{\zeta - \zeta^{-3}}{\lambda^4} \, d\zeta \, d\lambda \, dk; \tag{68}$$

see (20) and [2, (4.7)].

Lemma 3.7. Let Λ be a unitary character of $T(\mathbb{R})$. For $i=1,2$ let $l_i \ge l_i' > 0$ be integers. Let $B_{l_i,l_i'} \in \mathcal{S}(\Lambda,\theta,l_i,l_i')$, and let $B_i \in \mathcal{U}(\mathfrak{g}_\mathbb{C})B_{l_i,l_i'}$. If $l_1'+l_2'>4$, then

$$\int_{R(\mathbb{R})\backslash \mathrm{GSp}(4,\mathbb{R})} B_1(g)\overline{B_2(g)}\,dg \tag{69}$$

is absolutely convergent. □

Proof. By Lemma 3.5, we may assume that

$$B_i(h(\lambda,\zeta,0,0)k) = \left(\sum_{j=-(l_i-l_i')}^{l_i-l_i'} A_{i,j}(k)\zeta^j\right)(\zeta^2+\zeta^{-2})^{e_i}(\zeta^2-\zeta^{-2})^{f_i}P_i(\lambda)\lambda^{e_i+f_i+\frac{l_i+l_i'+s}{2}}e^{-2\pi\lambda(\zeta^2+\zeta^{-2})}, \tag{70}$$

where $A_{i,j}$ are analytic functions on K^1, and where P_i is a polynomial. Since $s\in i\mathbb{R}$ and we are taking absolute values, we may assume that $s=0$. Furthermore, by (68), we may in fact assume that

$$B_i(h(\lambda,\zeta,0,0)k) = \zeta^{j_i}(\zeta^2+\zeta^{-2})^{e_i}(\zeta^2-\zeta^{-2})^{f_i}P_i(\lambda)\lambda^{e_i+f_i+\frac{l_i+l_i'}{2}}e^{-2\pi\lambda(\zeta^2+\zeta^{-2})}, \tag{71}$$

where $j_i \le l_i - l_i'$. The relevant integral is then

$$\int_0^\infty\int_1^\infty \zeta^{j_1+j_2}(\zeta^2+\zeta^{-2})^{e_1+e_2}(\zeta^2-\zeta^{-2})^{f_1+f_2}P_1(\lambda)\overline{P_2(\lambda)}\lambda^{e_1+f_1+e_2+f_2+\frac{l_1+l_1'+l_2+l_2'}{2}}$$
$$\times\, e^{-4\pi\lambda(\zeta^2+\zeta^{-2})}\frac{\zeta-\zeta^{-3}}{\lambda^4}\,d\zeta\,d\lambda.$$

By Lemma 3.6, this integral is finite if and only if

$$e_1+f_1+e_2+f_2+\frac{l_1+l_1'+l_2+l_2'}{2}-4>-1 \tag{72}$$

and

$$j_1+j_2-1+2(e_1+e_2+f_1+f_2+1) < 2\left(e_1+f_1+e_2+f_2+\frac{l_1+l_1'+l_2+l_2'}{2}-4\right)+1. \tag{73}$$

Assuming that $l_1'+l_2'>4$, condition (72) is certainly satisfied. Since $j_i \le l_i - l_i'$, condition (73) is also satisfied provided that $l_1'+l_2'>4$. This concludes the proof. ■

3.4.3 Irreducibility of lowest weight modules

We will apply Lemma 3.7 to the case where $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})B_{l_2,l_2'} \subset \mathcal{U}(\mathfrak{g}_{\mathbb{C}})B_{l_1,l_1'}$. In this case, we will show in the following lemma that we always have $l_1' + l_2' > 4$. It will be clear from the proof that only the case $l_1' = 1$ needs to be considered. The lemma is purely algebraic and has nothing to do with the fact that our functions have the Bessel transformation property.

Lemma 3.8. Let Λ be the character of $T(\mathbb{R})$ defined by a pair $(s, m) \in \mathbb{C} \times \mathbb{Z}$, as in (43). Let $l > 0$ be an integer. We assume that $l + 1 + m$ is even and that $|m| \leq l - 1$, so that $\mathcal{S}(\Lambda, \theta, l, 1)$ is one-dimensional; see Theorem 3.4. Let $B_0 \in \mathcal{S}(\Lambda, \theta, l, 1)$ be a nonzero element.

(i) If $l \geq 3$, then the vectors

$$X_+^{\alpha_1} P_{1+}^2 B_0, \quad X_+^{\alpha_2} P_{0+} B_0, \quad X_+^{\alpha_3} P_{1+} N_- B_0, \quad X_+^{\alpha_4} N_-^2 B_0, \tag{74}$$

$$X_+^{\alpha_5} P_{1+} B_0, \quad X_+^{\alpha_6} N_- B_0, \tag{75}$$

$$X_+^{\alpha_7} B_0, \tag{76}$$

where α_i are non-negative integers, are linearly independent. No linear combination of these vectors with a fixed weight is annihilated by all of X_-, P_{1-} and P_{0-}.

(ii) If $l = 2$, then the same statement as in i) holds with the last vector in (74) omitted.

(iii) If $l = 1$, then the same statement as in i) holds with the last two vectors in (74) and the last vector in (75) omitted. □

Proof. (i) We will repeatedly use the identity

$$X_- X_+^{\alpha} = X_+^{\alpha} X_- - \alpha X_+^{\alpha-1} Z - \alpha(\alpha - 1) X_+^{\alpha-1} \qquad (\alpha \geq 1) \tag{77}$$

in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. If any linear combination of the vectors in (74)–(76) is zero, then this linear combination must contain only vectors of the same weight. Since $X_- B_0 = 0$, and $\langle X_+, Z, X_- \rangle$ is an $\mathfrak{sl}(2)$-triple, it is clear (or follows easily by induction using (77)) that the vectors $X_+^{\alpha} B_0$, $\alpha \geq 0$, are linearly independent, and that none of these vectors except B_0 is annihilated by X_-. Consider a linear combination of the vectors in (75):

$$a X_+^{\alpha} P_{1+} B_0 + b X_+^{\alpha+1} N_- B_0 = 0. \tag{78}$$

If $\alpha = 0$, then we get $a = b = 0$ by applying P_{1-} to (78). Assume that $\alpha > 0$. Applying X_- to (78) and using (77), we get

$$-a\alpha(l+\alpha)X_+^{\alpha-1}P_{1+}B_0 + (a - b(\alpha+1)(l+\alpha-1))X_+^{\alpha}N_-B_0 = 0. \tag{79}$$

It follows that $a = b = 0$ by induction on α. This proves the linear independence of the vectors in (75). Assuming linear independence, the same calculations show that no linear combination of the vectors in (75) is annihilated by both X_- and P_{1-}. Finally we have to consider linear combinations of the vectors in (74). The method is the same as above: Writing down a linear combination of vectors of the same weight and applying X_-, P_{1-} and P_{0-}, we get conditions on the coefficients by induction on α. We omit the details.

(ii) and (iii) are proved similarly. Note that $N_-^2B_0 = 0$ for $l = 2$ and $N_-B_0 = 0$ for $l = 1$. ■

Proposition 3.9. Let Λ be the character of $T(\mathbb{R})$ defined by a pair $(s,m) \in \mathbb{C} \times \mathbb{Z}$, as in (43). Let $l \geq l' > 0$ be integers. We assume that $l + l' + m$ is even and that $|m| \leq l - l'$, so that $\mathcal{S}(\Lambda,\theta,l,l')$ is one-dimensional; see Theorem 3.4. Let B_0 be a function spanning this space. Then the $(\mathfrak{g},K)$-module $\mathcal{U}(\mathfrak{g}_\mathbb{C})B_0$ is irreducible. □

Proof. After applying a suitable twist, we may assume that $s = 0$ (see the considerations leading up to (15); observe that $(X.B)\breve{} = X.\tilde{B}$ for any of our differential operators X). Let $W \subset \mathcal{U}(\mathfrak{g}_\mathbb{C})B_0$ be a proper invariant subspace; we will obtain a contradiction. It is easy to see that W contains a weight vector $\tilde{B}_0$ that is annihilated by X_-, P_{1-}, P_{0-} and N_+. Let (l_2,l_2') be the weight of $\tilde{B}_0$. Since $\tilde{B}_0$ is a (Λ,θ)-Bessel function, $\tilde{B}_0$ spans the one-dimensional space $\mathcal{S}(\Lambda,\theta,l_2,l_2')$. We have

$$\mathcal{U}(\mathfrak{g}_\mathbb{C})\tilde{B}_0 \subset W \subsetneq \mathcal{U}(\mathfrak{g}_\mathbb{C})B_0.$$

Evidently, $l_2' \geq l'$. Since $X_-X_+^{\alpha}B_0 \neq 0$ for any $\alpha > 0$ (see (77)), we have in fact $l_2' > l'$. By Lemma 3.8, if $l' = 1$, then $l_2' \geq 4$. Hence, $l' + l_2' > 4$ in any case. Therefore, by Lemma 3.7,

$$\langle B_1, B_2\rangle := \int_{R(\mathbb{R})\backslash \mathrm{GSp}(4,\mathbb{R})} B_1(g)\overline{B_2(g)}\,dg$$

is absolutely convergent for all $B_1 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})B_0$ and $B_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})\tilde{B}_0$. We have $\langle X.B_1, B_2\rangle = \langle B_1, X.B_2\rangle$ for all $X \in \mathfrak{g}$. Let $V = \{B_1 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})B_0 : \langle B_1, B_2\rangle = 0 \text{ for all } B_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})\tilde{B}_0\}$. Then V

is a proper, invariant subspace of $\mathcal{U}(\mathfrak{g}_\mathbb{C})B_0$. Since $\mathcal{U}(\mathfrak{g}_\mathbb{C})\tilde{B}_0 = \mathcal{U}(\mathfrak{k}_\mathbb{C} + \mathfrak{p}_+)\tilde{B}_0$ does not contain the K^1-type (l, l'), we have $B_0 \in V$. Hence $\mathcal{U}(\mathfrak{g}_\mathbb{C})B_0 \subset V \subsetneq \mathcal{U}(\mathfrak{g}_\mathbb{C})B_0$, a contradiction. ■

3.4.4 The main result

We are now ready to state the main result about nonsplit Bessel models of lowest weight representations. Recall that if the character Λ of $T(\mathbb{R})$ is given by a pair $(s, m) \in \mathbb{C} \times \mathbb{Z}$ as in (43), then the lowest weight representation $\mathcal{E}_t(l, l')$ of $\mathrm{GSp}(4, \mathbb{R})$ can only have a (Λ, θ)-Bessel model if $t = s$; see the end of Section 2.6.

Theorem 3.10. Let Λ be the character of $T(\mathbb{R})$ defined by a pair $(s, m) \in \mathbb{C} \times \mathbb{Z}$, as in (43). Let $l \geq l' > 0$ be integers. Then the lowest weight module $\mathcal{E}_s(l, l')$ of $\mathrm{GSp}(4, \mathbb{R})$ has a (Λ, θ)-Bessel model as defined in Section 2.6 if and only if $l + l' + m$ is even and $|m| \leq l - l'$. If a Bessel model exists, then it is unique. The Bessel function B_0 representing the highest weight vector in the minimal K-type of $\mathcal{E}_s(l, l')$ is the function described in Theorem 3.4. The general form of other functions in the Bessel model of $\mathcal{E}_s(l, l')$ is described in Lemma 3.5. □

Proof. Assume that $\pi = \mathcal{E}_s(l, l')$ has a (Λ, θ)-Bessel model $\mathcal{B}_\pi(\Lambda, \theta)$. Let $B_0 \in \mathcal{B}_\pi(\Lambda, \theta)$ be a nonzero vector with the properties (2). Then B_0 is an element of the space $\mathcal{S}(\Lambda, \theta, l, l')$ defined in Section 3.3. It follows from Theorem 3.4 that $l + l' + m$ is even and $|m| \leq l - l'$ (note that the first condition simply expresses the compatibility of Λ with the central character of $\mathcal{E}(l, l')$). The one-dimensionality of $\mathcal{S}(\Lambda, \theta, l, l')$ stated in Theorem 3.4 implies the uniqueness of the space $\mathcal{B}_\pi(\Lambda, \theta)$.

Conversely, assume that $l + l' + m$ is even and $|m| \leq l - l'$. Let B_0 be a function spanning the space $\mathcal{S}(\Lambda, \theta, l, l')$; see Theorem 3.4. By Proposition 3.9, the $(\mathfrak{g}, K)$-module $\mathcal{U}(\mathfrak{g}_\mathbb{C})B_0$ is irreducible. Since it contains a vector with the properties (2), it follows that $\mathcal{U}(\mathfrak{g}_\mathbb{C})B_0 \cong \mathcal{E}_s(l, l')$. Hence $\mathcal{U}(\mathfrak{g}_\mathbb{C})B_0$ provides a (Λ, θ)-Bessel model for $\mathcal{E}_s(l, l')$. ■

Corollary 3.11. Let Λ be a character of $T^1(\mathbb{R}) \cong S_1$ defined by $m \in \mathbb{Z}$; see (43).

(i) Let $l \geq l' > 0$ be integers. Then the lowest weight module $\mathcal{E}(l, l')$ of $\mathrm{Sp}(4, \mathbb{R})$ has a (Λ, θ)-Bessel model as defined in Section 2.6 if and only if $l + l' + m$ is even and $|m| \leq l - l'$. If a Bessel model exists, then it is unique. The Bessel function B_0 representing the highest weight vector in the minimal K^1-type of $\mathcal{E}(l, l')$ is the restriction of the function described in Theorem 3.4 to $\mathrm{Sp}(4, \mathbb{R})$.

The general form of other functions in the Bessel model of $\mathcal{E}(l,l')$ is described in Lemma 3.5.

(ii) Let $l' \le l < 0$ be integers. Then the highest weight module $\mathcal{E}(l,l')$ of $\mathrm{Sp}(4,\mathbb{R})$ does not admit a (Λ,θ)-Bessel model. □

Proof. This follows from Theorem 3.10 and the considerations leading up to Proposition 2.2. ■

Note that we have throughout assumed that $S = \left[\begin{smallmatrix}1&\\&1\end{smallmatrix}\right]$ is the matrix defining the character θ in (4). If we would have chosen $S = \left[\begin{smallmatrix}-1&\\&-1\end{smallmatrix}\right]$, then it would be the *highest* weight representations of $\mathrm{Sp}(4,\mathbb{R})$ that admit Bessel models; see (14).

3.4.5 Realization in L^p-spaces

As a consequence of our explicit formulas for the Bessel functions, we show that the Bessel models for most lowest-weight representations lie in certain L^p-spaces.

Lemma 3.12. Let Λ be the character of $T(\mathbb{R})$ defined by a pair $(s,m) \in \mathbb{C}\times\mathbb{Z}$, as in (43). Assume that $s \in i\mathbb{R}$, so that Λ is a unitary character. Assume further that $l+l'+m$ is even and that $|m| \le l-l'$, so that $\mathcal{S}(\Lambda,\theta,l,l')$ is a one-dimensional space; see Theorem 3.4. Let B_0 be a function spanning this space.

(i) For any $\varepsilon > 0$,

$$\int_{R(\mathbb{R})\backslash \mathrm{GSp}(4,\mathbb{R})} |B_0(x)|^{2+\varepsilon}\,dx < \infty \iff l' \ge 2. \tag{80}$$

(ii) We have

$$\int_{R(\mathbb{R})\backslash \mathrm{GSp}(4,\mathbb{R})} |B_0(x)|^{2}\,dx < \infty \iff l' \ge 3. \tag{81}$$

□

Proof. Using the formula for B_0 from Theorem 3.4, this can be proved similarly as Lemma 3.6. ■

Assuming that the character Λ is unitary and $p > 0$, let $L^p(R(\mathbb{R})\backslash \mathrm{GSp}(4,\mathbb{R}),\Lambda,\theta)$ be the space of all measurable functions $B:\ \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ with the Bessel transformation property $B(tug) = \Lambda(t)\theta(u)B(g)$ for $t \in T(\mathbb{R})$ and $u \in U(\mathbb{R})$, and such that

$$\int_{R(\mathbb{R})\backslash \mathrm{GSp}(4,\mathbb{R})} |B(g)|^p \, dg < \infty. \tag{82}$$

Proposition 3.13. Let Λ be the character of $T(\mathbb{R})$ defined by a pair $(s, m) \in \mathbb{C} \times \mathbb{Z}$, as in (43). Assume that $s \in i\mathbb{R}$, so that Λ is a unitary character. Let $l \geq l' > 0$ be integers. We assume that $l + l' + m$ is even and that $|m| \leq l - l'$, so that the lowest weight $(\mathfrak{g}, K)$-module $\mathcal{E}_s(l, l')$ possesses a (Λ, θ)-Bessel model.

(i) Let $\varepsilon > 0$ be arbitrary. Then the (Λ, θ)-Bessel model for $\mathcal{E}_s(l, l')$ lies in $L^{2+\varepsilon}(R(\mathbb{R})\backslash \mathrm{GSp}(4, \mathbb{R}), \Lambda, \theta)$ if and only if $l' \geq 2$.

(ii) The (Λ, θ)-Bessel model for $\mathcal{E}_s(l, l')$ lies in $L^2(R(\mathbb{R})\backslash \mathrm{GSp}(4, \mathbb{R}), \Lambda, \theta)$ if and only if $l' \geq 3$.

□

Proof. (i) By Lemma 3.12, the lowest weight vector $B_0 \in \mathcal{S}(\Lambda, \theta, l, l')$ lies in $L^{2+\varepsilon}(R(\mathbb{R})\backslash \mathrm{GSp}(4, \mathbb{R}), \Lambda, \theta)$ if and only if $l' \geq 2$. Since the Bessel model of $\mathcal{E}_s(l, l')$ is generated by right translates of B_0 and right translation preserves the $L^{2+\varepsilon}$ norm, we have $\mathcal{E}_s(l, l') \subset L^{2+\varepsilon}(R(\mathbb{R})\backslash \mathrm{GSp}(4, \mathbb{R}), \Lambda, \theta)$ if and only if $B_0 \in L^{2+\varepsilon}(R(\mathbb{R})\backslash \mathrm{GSp}(4, \mathbb{R}), \Lambda, \theta)$. This proves (i).

(ii) is proved in the same way. ■

Remark. This result is plausible, given that $\mathcal{E}(l, l')$ is a discrete series representation for $l' \geq 3$, and a limit of discrete series representation for $l' = 2$. □

4 Split Bessel Models

In this section, we investigate the existence and uniqueness of split Bessel models for the lowest and highest weight representations of GSp(4, ℝ) and Sp(4, ℝ). As in the nonsplit case, we shall work with GSp(4, ℝ) and use the discussion preceding Proposition 2.2 to obtain results for Sp(4, ℝ). After changing models, as explained in Section 2.6, we may throughout assume that

$$S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

4.1 Double coset decomposition

Again we start by deriving representatives for the double coset space $R(\mathbb{R})\backslash\mathrm{GSp}(4,\mathbb{R})/K^1$, where $R(\mathbb{R}) = T(\mathbb{R})_0 U(\mathbb{R})$. Let $S = [\begin{smallmatrix}1&\\&-1\end{smallmatrix}]$. In this case,

$$T(\mathbb{R}) = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} : \; x, y \in \mathbb{R}, \; x^2 - y^2 \neq 0 \right\}. \tag{83}$$

Let $t_0 = [\begin{smallmatrix}1&1\\1&-1\end{smallmatrix}]$ and let $A := \{[\begin{smallmatrix}a&0\\0&b\end{smallmatrix}] : \; a, b \in \mathbb{R}^\times\}$ be the split torus in $\mathrm{GL}(2,\mathbb{R})$. We have $T(\mathbb{R}) \simeq A$ via the map

$$T(\mathbb{R}) \ni \begin{bmatrix} x & y \\ y & x \end{bmatrix} \longmapsto t_0^{-1} \begin{bmatrix} x & y \\ y & x \end{bmatrix} t_0 = \begin{bmatrix} x+y & 0 \\ 0 & x-y \end{bmatrix} \in A. \tag{84}$$

Let $N := \{[\begin{smallmatrix}1&\zeta\\0&1\end{smallmatrix}] : \; \zeta \in \mathbb{R}\}$. The Iwasawa decomposition for $\mathrm{GL}(2,\mathbb{R})$ implies

$$\mathrm{GL}(2,\mathbb{R}) = A \cdot N \cdot \mathrm{SO}(2) = t_0 A \cdot N \cdot \mathrm{SO}(2) = t_0 A t_0^{-1} \cdot t_0 N \cdot \mathrm{SO}(2) = T(\mathbb{R}) \cdot t_0 N \cdot \mathrm{SO}(2). \tag{85}$$

Using this and the Iwasawa decomposition for $\mathrm{GSp}(4,\mathbb{R})$, we get

$$\mathrm{GSp}(4,\mathbb{R}) = T(\mathbb{R})U(\mathbb{R}) \left\{ \begin{bmatrix} \lambda t_0 \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix} & \\ & \det(t_0)^t t_0^{-1} \begin{bmatrix} 1 & 0 \\ -\zeta & 1 \end{bmatrix} \end{bmatrix} : (\lambda, \zeta) \in \mathbb{R}^\times \times \mathbb{R} \right\} K^1. \tag{86}$$

Any signs appearing in $T(\mathbb{R}) \cong \mathbb{R}^\times \times \mathbb{R}^\times$ can be absorbed into λ and K^1, so that $T(\mathbb{R})$ can be replaced by $T(\mathbb{R})_0$. Hence,

$$\mathrm{GSp}(4,\mathbb{R}) = R(\mathbb{R}) \left\{ \begin{bmatrix} \lambda t_0 \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix} & \\ & \det(t_0)^t t_0^{-1} \begin{bmatrix} 1 & 0 \\ -\zeta & 1 \end{bmatrix} \end{bmatrix} : (\lambda, \zeta) \in \mathbb{R}^\times \times \mathbb{R} \right\} K^1. \tag{87}$$

It can be checked that the double cosets in (87) are disjoint. Recalling the coordinates (1) in a neighborhood of the identity of K^1, we let

$$\hat{h}(\lambda,\zeta,\varphi_1,\varphi_2) := \begin{bmatrix} \lambda t_0 \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix} & \\ & \det(t_0)^t t_0^{-1} \begin{bmatrix} 1 & 0 \\ -\zeta & 1 \end{bmatrix} \end{bmatrix} r_1(\varphi_1) r_2(\varphi_2). \tag{88}$$

4.2 Differential operators

Following the method of Section 3.2, we will now derive explicit formulas for the differential operators given by elements of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on the functions in a split Bessel model. Assume that $\mathcal{B}_{\Lambda,\theta}(\pi)$ is a Bessel model for the $(\mathfrak{g}, K)$-module (π, V). For any $B \in \mathcal{B}_{\Lambda,\theta}(\pi)$, we define a function $f = f_B$ on $\mathbb{R}^\times \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by

$$f(\lambda,\zeta,\varphi_1,\varphi_2) = B(\hat{h}(\lambda,\zeta,\varphi_1,\varphi_2)). \tag{89}$$

It follows from (87) that if B has a weight (l, l'), then B is determined by f. If L denotes one of the operators $N_\pm$, $X_\pm$, $P_{0\pm}$, $P_{1\pm}$, then $L.B$ will be determined by the associated function $f_{L.B}$. We first calculate the action of an element L of the noncomplexified Lie algebra $\mathfrak{g}$, given by

$$(L.B)(\hat{h}(\lambda,\zeta,\varphi_1,\varphi_2)) = \frac{d}{dt}\Big|_0 B(\hat{h}(\lambda,\zeta,\varphi_1,\varphi_2)\exp(tL)).$$

At least for small values of t, we can decompose the argument according to (87),

$$\hat{h}(\lambda,\zeta,\varphi_1,\varphi_2)\exp(tL) = \begin{bmatrix} 1 & x(t) & y(t) \\ & 1 & y(t) & z(t) \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} g(t) & \\ & \det(g(t))\,{}^t g(t)^{-1} \end{bmatrix}$$
$$\hat{h}(\lambda(t),\zeta(t),\varphi_1(t),\varphi_2(t))\, r_3(\varphi_3(t)) r_4(\varphi_4(t)). \tag{90}$$

Here, $g(t) \in T(\mathbb{R})_0$, and $x(t)$ etc. are smooth functions in a neighborhood of 0 satisfying

$$x(0) = y(0) = z(0) = \varphi_3(0) = \varphi_4(0) = 0,$$
$$\lambda(0) = \lambda, \quad \zeta(0) = \zeta, \quad \varphi_1(0) = \varphi_1, \quad \varphi_2(0) = \varphi_2.$$

According to (84), we can write

$$g(t) = t_0 \begin{bmatrix} a(t) & 0 \\ 0 & b(t) \end{bmatrix} t_0^{-1} \tag{91}$$

with smooth functions $a(t) > 0$ and $b(t) > 0$ such that $a(0) = 1$ and $b(0) = 1$. The character Λ of $T(\mathbb{R})_0$ is of the form

$$\Lambda\left(t_0 \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} t_0^{-1}\right) = a^{s_1} b^{s_2}, \qquad a, b > 0, \tag{92}$$

with some $s_1, s_2 \in \mathbb{C}$. It follows that

$$\begin{aligned}
&(L.B)(\hat{h}(\lambda, \zeta, \varphi_1, \varphi_2)) \\
&= \frac{d}{dt}\Big|_0 \left(\theta\left(\begin{bmatrix} 1 & x(t) & y(t) & \\ & 1 & y(t) & z(t) \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \Lambda(g(t)) e^{i(l\varphi_3(t)+l'\varphi_4(t))} B(\hat{h}(\lambda(t), \zeta(t), \varphi_1(t), \varphi_2(t)))\right) \\
&= \frac{d}{dt}\Big|_0 \left(a(t)^{s_1} b(t)^{s_2} e^{2\pi i(x(t)-z(t))} e^{i(l\varphi_3(t)+l'\varphi_4(t))} f(\lambda(t), \zeta(t), \varphi_1(t), \varphi_2(t))\right) \\
&= (s_1 a'(0) + s_2 b'(0) + i(l\varphi_3'(0) + l'\varphi_4'(0)) + 2\pi i(x'(0) - z'(0)))\, f(\lambda, \zeta, \varphi_1, \varphi_2) + \lambda'(0) \frac{\partial f}{\partial \lambda}(\lambda, \zeta, \varphi_1, \varphi_2) \\
&\quad + \zeta'(0) \frac{\partial f}{\partial \zeta}(\lambda, \zeta, \varphi_1, \varphi_2) + \varphi_1'(0) \frac{\partial f}{\partial \varphi_1}(\lambda, \zeta, \varphi_1, \varphi_2) + \varphi_2'(0) \frac{\partial f}{\partial \varphi_2}(\lambda, \zeta, \varphi_1, \varphi_2).
\end{aligned} \tag{93}$$

Thus we need the derivatives at 0 of the auxiliary functions $\lambda, \zeta, \ldots$. To get these, we differentiate the matrix equation (90) and put $t = 0$. This yields 16 linear equations from which the desired derivatives can be determined. We will refrain from listing all these derivatives, and instead just state the resulting formulas for the action of the complexified Lie algebra. Let us write h for the element $\hat{h}(\lambda, \zeta, \varphi_1, \varphi_2)$.

$$Z.B = lB, \tag{94}$$

$$Z'.B = l'B, \tag{95}$$

$$(N_\pm.B)(h) = \frac{i}{2}\tan(2\varphi_2)(l' - l)\, f(\lambda, \zeta, \varphi_1, \varphi_2) + \frac{1}{2\cos(2\varphi_2)} \frac{\partial f}{\partial \varphi_1}(\lambda, \zeta, \varphi_1, \varphi_2), (\lambda, \zeta, \varphi_1, \varphi_2) \mp \frac{i}{2} \frac{\partial f}{\partial \varphi_2} \tag{96}$$

$$\begin{aligned}(X_{\pm}.B)(h) = &\Big(\frac{1}{4}(s_1 - s_2)(\cos(2\varphi_1) \pm i\sin(2\varphi_1)\sin(2\varphi_2)) - \frac{1}{4}(s_1+s_2)\cos(2\varphi_2)\\ &\pm \frac{l(\cos^4(\varphi_2)+\sin^4(\varphi_2))}{2\cos(2\varphi_2)} \mp \frac{l'}{4}\sin(2\varphi_2)\tan(2\varphi_2) + 2\pi i\lambda((\cos(2\varphi_1)\\ &-\zeta\sin(2\varphi_1))\sin(2\varphi_2) \pm i(\zeta\cos(2\varphi_1) - \zeta\cos(2\varphi_2) + \sin(2\varphi_1)))\Big) f(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\frac{1}{2}\cos(2\varphi_2)\lambda\frac{\partial f}{\partial\lambda}(\lambda,\zeta,\varphi_1,\varphi_2) \qquad (97)\\ &+\frac{1}{2}(-\zeta\cos(2\varphi_1) - \sin(2\varphi_1) \pm i\sin(2\varphi_2)(\cos(2\varphi_1)-\zeta\sin(2\varphi_1)))\frac{\partial f}{\partial\zeta}(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\left(\frac{1}{4}\sin(2\varphi_1) \mp \frac{i}{4}(\cos(2\varphi_1)\cos(2\varphi_2)-1)\tan(2\varphi_2)\right)\frac{\partial f}{\partial\varphi_1}(\lambda,\zeta,\varphi_1,\varphi_2) \qquad (98)\\ &+\frac{1}{4}\sin(2\varphi_2)\frac{\partial f}{\partial\varphi_2}(\lambda,\zeta,\varphi_1,\varphi_2), \qquad (99)\end{aligned}$$

$$\begin{aligned}(P_{1\pm}.B)(h) = &\Big(\frac{1}{2}(s_1-s_2)\sin(2\varphi_1)\cos(2\varphi_2) \mp \frac{i}{2}(s_1+s_2)\sin(2\varphi_2) + \frac{i}{2}\sin(2\varphi_2)(l+l')\\ &-4\pi i\lambda(\pm i\cos(2\varphi_2)(\cos(2\varphi_1)-\zeta\sin(2\varphi_1)) - \zeta\sin(2\varphi_2))\Big) f(\lambda,\zeta,\varphi_1,\varphi_2)\\ &\pm i\sin(2\varphi_2)\lambda\frac{\partial f}{\partial\lambda}(\lambda,\zeta,\varphi_1,\varphi_2) + \cos(2\varphi_2)(\cos(2\varphi_1)-\zeta\sin(2\varphi_1))\frac{\partial f}{\partial\zeta}(\lambda,\zeta,\varphi_1,\varphi_2)\\ &-\frac{1}{2}\cos(2\varphi_1)\cos(2\varphi_2)\frac{\partial f}{\partial\varphi_1}(\lambda,\zeta,\varphi_1,\varphi_2) \mp \frac{i}{2}\cos(2\varphi_2)\frac{\partial f}{\partial\varphi_2}(\lambda,\zeta,\varphi_1,\varphi_2), \qquad (100)\end{aligned}$$

$$\begin{aligned}(P_{0\pm}.B)(h) = &\Big(\frac{1}{4}(s_1-s_2)(-\cos(2\varphi_1) \pm i\sin(2\varphi_1)\sin(2\varphi_2)) - \frac{1}{4}(s_1+s_2)\cos(2\varphi_2)\\ &\mp\frac{l}{4}\sin(2\varphi_2)\tan(2\varphi_2) \pm \frac{l'(\sin^4(\varphi_2)+\cos^4(\varphi_2))}{2\cos(2\varphi_2)} + 2\pi i\lambda((\cos(2\varphi_1)\\ &-\zeta\sin(2\varphi_1))\sin(2\varphi_2) \mp i(\zeta\cos(2\varphi_1) + \zeta\cos(2\varphi_2) + \sin(2\varphi_1)))\Big) f(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\frac{1}{2}\cos(2\varphi_2)\lambda\frac{\partial f}{\partial\lambda}(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\frac{1}{2}(\zeta\cos(2\varphi_1) + \sin(2\varphi_1) \pm i(\cos(2\varphi_1)-\zeta\sin(2\varphi_1))\sin(2\varphi_2))\frac{\partial f}{\partial\zeta}(\lambda,\zeta,\varphi_1,\varphi_2)\\ &-\left(\frac{1}{4}\sin(2\varphi_1) \pm \frac{i}{4}(1+\cos(2\varphi_1)\cos(2\varphi_2))\tan(2\varphi_2)\right)\frac{\partial f}{\partial\varphi_1}(\lambda,\zeta,\varphi_1,\varphi_2)\\ &+\frac{1}{4}\sin(2\varphi_2)\frac{\partial f}{\partial\varphi_2}(\lambda,\zeta,\varphi_1,\varphi_2). \qquad (101)\end{aligned}$$

4.3 Nonexistence of split Bessel models

In this section, we will show that the lowest weight representations of GSp(4, ℝ) do not admit split Bessel models. Let Λ be the character of $T(\mathbb{R})_0$ defined in (92). Let $l \geq l' > 0$

be integers, and let $\pi = \mathcal{E}(l, l')$ be a lowest weight representation of $\mathrm{GSp}(4,\mathbb{R})$ as defined in Sections 2.3 and 2.4. As in Section 3.3, let $\mathcal{S}(\Lambda, \theta, l, l')$ be the space of all functions $B : \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ satisfying the following conditions.

($\mathcal{S}1$) B is smooth and K-finite.

($\mathcal{S}2$) $B(tug) = \Lambda(t)\theta(u)B(g)$ for all $t \in T(\mathbb{R})_0$, $u \in U(\mathbb{R})$, $g \in \mathrm{GSp}(4,\mathbb{R})$.

($\mathcal{S}3$) B is slowly increasing.

($\mathcal{S}4$) $Z.B = lB$ and $Z'.B = l'B$. Equivalently, $B(gr_3(\varphi_3)r_4(\varphi_4)) = e^{i(l\varphi_3 + l'\varphi_4)}B(g)$ for all $\varphi_3, \varphi_4 \in \mathbb{R}$, $g \in \mathrm{GSp}(4,\mathbb{R})$.

($\mathcal{S}5$) $N_+.B = X_-.B = P_{1-}.B = P_{0-}.B = 0$.

If B is a highest weight vector in the minimal K-type in a Bessel model for π of type (Λ, θ), then B satisfies ($\mathcal{S}1$)–($\mathcal{S}5$). Given such a B, we define the associated function f as in (89). Note that f is an analytic function on $\mathbb{R}^\times \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Calculations using formulas (96)–(101) show that condition ($\mathcal{S}5$) is equivalent to the following system of differential equations:

$$\frac{\partial f}{\partial \lambda} = \left(\frac{l + l' + s_1 + s_2}{2\lambda} + 4\pi\zeta\right) f, \tag{102}$$

$$\frac{\partial f}{\partial \zeta} = \left(4\pi\lambda - \frac{(l - l' + s_1 - s_2)(\cos(\varphi_2)\sin(\varphi_1) - i\cos(\varphi_1)\sin(\varphi_2))}{2\cos(\varphi_2)(\cos(\varphi_1) - \zeta\sin(\varphi_1)) + 2i(\zeta\cos(\varphi_1) + \sin(\varphi_1))\sin(\varphi_2)}\right) f, \tag{103}$$

$$\frac{\partial f}{\partial \varphi_1} = \left(\frac{(s_1 - s_2)\cos(2\varphi_2) + (l - l')(\zeta\sin(2\varphi_1) - \cos(2\varphi_1) - i\sin(2\varphi_2)(\sin(2\varphi_1) + \zeta\cos(2\varphi_1))}{\zeta\cos(2\varphi_2) + \cos(2\varphi_1)(-\zeta + i\sin(2\varphi_2)) - \sin(2\varphi_1)(1 + i\zeta\sin(2\varphi_2))}\right) f, \tag{104}$$

$$\frac{\partial f}{\partial \varphi_2} = \left(\frac{-i(s_1 - s_2) + (l - l')(-\zeta\sin(2\varphi_2) + i\cos(2\varphi_2)(\cos(2\varphi_1) - \zeta\sin(2\varphi_1)))}{\zeta\cos(2\varphi_2) + \cos(2\varphi_1)(-\zeta + i\sin(2\varphi_2)) - \sin(2\varphi_1)(1 + i\zeta\sin(2\varphi_2))}\right) f. \tag{105}$$

From (102) we get

$$f(\lambda, \zeta, \varphi_1, \varphi_2) = \begin{cases} c_1(\zeta, \varphi_1, \varphi_2)\lambda^{\frac{l+l'+s_1+s_2}{2}} e^{4\pi\lambda\zeta} & \text{if } \lambda > 0, \\ c_2(\zeta, \varphi_1, \varphi_2)(-\lambda)^{\frac{l+l'+s_1+s_2}{2}} e^{4\pi\lambda\zeta} & \text{if } \lambda < 0, \end{cases} \tag{106}$$

with certain functions $c_1(\zeta, \varphi_1, \varphi_2)$ and $c_2(\zeta, \varphi_1, \varphi_2)$. Note that c_1 and c_2 are analytic functions on all of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Assume that c_1 is not constantly zero. Then, by analyticity, there exists a choice of $(\zeta, \varphi_1, \varphi_2) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$ such that $c_1(\zeta, \varphi_1, \varphi_2) \neq 0$. But a look at (106) shows that this would violate the moderate growth condition for B; see (13). Similarly, the assumption that c_2 is not constantly zero also violates moderate growth. This proves the following.

Theorem 4.1. Let $S \in M(2 \times 2, \mathbb{R})$ be a nondegenerate symmetric matrix with $\det(S) < 0$. Let θ be the corresponding character of $U(\mathbb{R})$ as in (4), and let $T(\mathbb{R})$ be the group defined in (5). Let $l \geq l' > 0$ be integers. Then, for any character Λ of $T(\mathbb{R})_0 \cong \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, the space $\mathcal{S}(\Lambda, \theta, l, l')$ is zero. □

Corollary 4.2. The lowest weight representations $\mathcal{E}(l, l')$ of GSp(4, ℝ) do not admit split Bessel models. The lowest and highest weight representations of Sp(4, ℝ) also do not admit split Bessel models. □

Proof. The highest weight vector in the minimal K-type of a split Bessel model for $\mathcal{E}(l, l')$ would be a nonzero element of $\mathcal{S}(\Lambda, \theta, l, l')$. The assertion about Sp(4, ℝ) follows from Proposition 2.2. ■

Remark. The system (102)–(105) can be solved formally. Restricting to the connected component of the domain of f given by $\lambda > 0$, the one-dimensional solution space is spanned by the function

$$f(\lambda, \zeta, \varphi_1, \varphi_2) = (\cos(\varphi_2)(\cos(\varphi_1) - \zeta \sin(\varphi_1)) + i(\zeta \cos(\varphi_1) + \sin(\varphi_1)) \sin(\varphi_2))^{\frac{l-l'+s_1-s_2}{2}} \times (\sin(\varphi_1) \cos(\varphi_2) - i \cos(\varphi_1) \sin(\varphi_2))^{\frac{l-l'-s_1+s_2}{2}} \lambda^{\frac{l+l'+s_1+s_2}{2}} e^{4\pi\lambda\zeta}. \quad (107)$$

□

5 An Application

In the previous sections we obtained the formula for the highest weight vector in the minimal K-type of a lowest weight representation. One of the main uses for such a formula is for explicit computations involving the archimedean Bessel models. For example, if F is a scalar valued Siegel modular form and f is a Maaß form, then the formula for the archimedean Bessel function (already obtained in [8]) was used in [2] and [6] to obtain an integral representation of the degree-8 L-function $L(s, F \times f)$.

Since we now have the formula for the Bessel function for any lowest weight representation, and in particular any holomorphic discrete series representation, we will use it to obtain an integral representation for $L(s, \mathbf{F} \times f)$, where $\mathbf{F}$ is a vector-valued holomorphic Siegel modular form. The vector entering the archimedean zeta integral will actually *not* directly correspond to the modular form $\mathbf{F}$, but will be a vector spanning a certain one-dimensional K-type in the lowest weight representation generated by $\mathbf{F}$. We will give an algorithm to obtain the formula for such a vector and explicitly compute it in some cases. Then we will briefly recall vector-valued Siegel modular forms and the Bessel models associated with them. Finally, we will consider the integral representation of Furusawa and compute the archimedean integral in the vector-valued holomorphic Siegel modular forms case.

5.1 Finding good vectors

Let $l \geq l' > 0$ be integers of the same parity, and consider the lowest weight representation $\mathcal{E}_s(l,l')$ of $\mathrm{GSp}(4,\mathbb{R})$. In earlier sections, we have obtained a formula for the highest weight vector in the minimal K-type for such representations. In this section, we give an algorithm (and some examples) to find the formula for the vector in the one-dimensional K-type (l,l). Let B_0 be the lowest weight vector in the (Λ,θ)-Bessel model of $\mathcal{E}_s(l,l')$, as described in Theorem 3.4. We denote this function also by $B_{l,l'}$, and define recursively for $k = 1,\dots,(l-l')/2$

$$B_{l,l'+2k} = \left(P_{0+} + \frac{1}{\alpha}N_-N_+P_{0+} + \frac{1}{2\alpha(\alpha+1)}N_-^2N_+^2P_{0+}\right)B_{l,l'+2k-2}, \quad \alpha = l-l'-2k+2. \tag{108}$$

Calculations using the multiplication table given in Section 2.1 show that $N_+B_{l,l'+2k} = 0$ for all k (use that $N_+^3P_{0+}v = 0$ for highest weight vectors v). Hence, $B_{l,l'+2k}$ is the highest weight vector in the K-type $(l,l'+2k)$. Let us now assume that $S = \left[\begin{smallmatrix}1&\\&1\end{smallmatrix}\right]$ and $\Lambda \equiv 1$. Using the formula for $B_{l,l'}$ from Theorem 3.4 and (108), we will give the formula for $B_{l,l}$ in a few cases. Note that, since $B_{l,l}$ lies in a one-dimensional K-type, it is completely determined by its values on $h(\lambda,\zeta,0,0)$ (see (21)). Set $x = (\zeta^2+\zeta^{-2})/2$. For $\lambda < 0$ we have $B_{l,l}(h(\lambda,\zeta,0,0)) = 0$. For $\lambda > 0$ we have the following formulas:

$l' = l-2$:

$$B_{l,l}(h(\lambda,\zeta,0,0)) = 4e^{-4\pi\lambda x}\lambda^{l-1}(2(l-3)x + 8\pi\lambda); \tag{109}$$

$l' = l-4$:

$$B_{l,l}(h(\lambda,\zeta,0,0)) = \frac{4}{15}e^{-4\pi\lambda x}\lambda^{l-2}(12(l-4)(l-5)x^2 - 8(l-4)x(8\pi\lambda) + 2(8\pi\lambda)^2 - 4(l-4)(l-5)); \tag{110}$$

$l' = l-6$:

$$\begin{aligned} B_{l,l}(h(\lambda,\zeta,0,0)) = \frac{8}{35}e^{-4\pi\lambda x}\lambda^{l-3}&(40(l-5)(l-6)(l-7)x^3 - 36(l-5)(l-6)(8\pi\lambda)x^2 \\ &+ 12(l-5)(8\pi\lambda)^2x - 2(8\pi\lambda)^3 - 24(l-5)(l-6)(l-7)x + 12(l-5)(l-6)(8\pi\lambda)); \end{aligned} \tag{111}$$

$l' = l - 8$:

$$\begin{aligned} B_{l,l}(h(\lambda,\zeta,0,0)) = \frac{16}{315} e^{-4\pi\lambda x}\lambda^{l-4}(&560(l-6)(l-7)(l-8)(l-9)x^4 - 640(l-6)(l-7)(l-8)(8\pi\lambda)x^3 \\ &+ 288(l-6)(l-7)(8\pi\lambda)^2x^2 - 64(l-6)(8\pi\lambda)^3x + 8(8\pi\lambda)^4 \\ &- 480(l-6)(l-7)(l-8)(l-9)x^2 + 384(l-6)(l-7)(l-8)(8\pi\lambda)x \\ &- 96(l-6)(l-7)(8\pi\lambda)^2 + 48(l-6)(l-7)(l-8)(l-9)). \end{aligned} \tag{112}$$

5.2 Vector-valued Siegel modular forms and global Bessel models

Let $\mathfrak{h}_2 := \{Z \in M_2(\mathbb{C}) : {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}$ be the Siegel upper half plane of degree 2. Let $\Gamma_2 = \mathrm{Sp}(4,\mathbb{Z})$. Let n be an odd, positive integer. Let (ρ_0, V) be the polynomial (holomorphic), irreducible, n-dimensional representation of $\mathrm{GL}(2,\mathbb{C})$ for which the center acts trivially. For a positive integer $l \geq n$, let us denote by ρ the representation of $\mathrm{GL}(2,\mathbb{C})$ on V given by $g \mapsto \det(g)^l\rho_0(g)$. A vector-valued Siegel modular form of type ρ is defined as a holomorphic function $\mathbf{F} : \mathfrak{h}_2 \to V$ satisfying

$$\mathbf{F}(\gamma\langle Z\rangle) = \rho(CZ+D)\mathbf{F}(Z), \quad \text{where } \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_2,\ Z \in \mathfrak{h}_2,\ \gamma\langle Z\rangle := (AZ+B)(CZ+D)^{-1}. \tag{113}$$

Such a function has a Fourier expansion of the form

$$\mathbf{F}(Z) = \sum_{S \geq 0} \mathbf{A}(S)e^{2\pi i \operatorname{tr}(SZ)}, \tag{114}$$

where S runs through all semi-integral, semi-positive definite, symmetric 2×2 matrices. We say that $\mathbf{F}$ is a cusp form if $\mathbf{A}(S) \neq 0$ only if $S > 0$. We denote the space of vector-valued Siegel cusp forms of type ρ with respect to Γ_2 by $S_\rho(\Gamma_2)$. Let us assume that $\mathbf{F} \in S_\rho(\Gamma_2)$ is a Hecke eigenform. We will now construct the automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$ corresponding to $\mathbf{F}$. For $g = g_{\mathbb{Q}}g_\infty k_0$, with $g_{\mathbb{Q}} \in \mathrm{GSp}(4,\mathbb{Q})$, $g_\infty = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{GSp}(4,\mathbb{R})^+$ and $k_0 \in \prod_{p<\infty} \mathrm{GSp}(4,\mathbb{Z}_p)$, define

$$\mathbf{\Phi}(g) := \mu_2(g_\infty)^{l+(n-1)/2}\rho(CI+D)^{-1}\mathbf{F}(g_\infty\langle I\rangle), \tag{115}$$

where $I = [\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}]$. Choose a fixed nonzero linear functional Ψ on V, and set

$$\Phi(g) := \Psi(\mathbf{\Phi}(g)), \qquad g \in \mathrm{GSp}(4,\mathbb{A}). \tag{116}$$

Let $(\pi_{\mathbf{F}}, V_{\mathbf{F}})$ be an irreducible subspace of the $\mathrm{GSp}(4,\mathbb{A})$-space obtained from right translates of Φ. Then $\pi_{\mathbf{F}}$ is an irreducible, cuspidal, automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$. Note that $\pi_{\mathbf{F}}$ does not depend on the choice of Ψ. If $\pi_{\mathbf{F}} = \otimes' \pi_p$, then, for $p < \infty$, π_p is an unramified representation of $\mathrm{GSp}(4,\mathbb{Q}_p)$, and π_∞ is the lowest weight representation $\mathcal{E}(l, l-(n-1))$ (which is a holomorphic discrete series representation if $l \geq n+2$).

Let S be a positive definite, semi-integral, symmetric 2×2 matrix. Let the discriminant of S be given by $d(S) = -\det(2S) = -D$ and $L = \mathbb{Q}(\sqrt{-D})$. Let $T(\mathbb{A}) \simeq \mathbb{A}_L^\times$ be the adelic points of the group defined in (5). Let $R(\mathbb{A}) = T(\mathbb{A})U(\mathbb{A})$ be the Bessel subgroup of $\mathrm{GSp}(4,\mathbb{A})$. Let Λ be an ideal class character of L, i.e. a character of

$$T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R}) \prod_{p<\infty} (T(\mathbb{Q}_p) \cap \mathrm{GL}(2,\mathbb{Z}_p)).$$

Let ψ be a character of $\mathbb{Q}\backslash\mathbb{A}$ that is trivial on $\mathbb{Z}_p$ for all primes p and satisfies $\psi(x) = e^{-2\pi i x}$ for all $x \in \mathbb{R}$.

We define the global Bessel function of type $(\bar{\Lambda}, -S, \psi)$ associated to $\bar{\phi} \in V_{\mathbf{F}}$ by

$$B_{\bar{\phi}}(g) = \int_{Z(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})} (\Lambda \otimes \theta)(r)^{-1} \bar{\phi}(rg)\, dr, \tag{117}$$

where $\theta(\left[\begin{smallmatrix}1 & X\\ & 1\end{smallmatrix}\right]) = \psi(\mathrm{tr}(SX))$, $Z(\mathbb{A})$ is the center of $\mathrm{GSp}(4,\mathbb{A})$ and $\bar{\phi}(h) = \overline{\phi(h)}$. Note that $\pi_{\mathbf{F}}$ has a global Bessel model of type $(\bar{\Lambda}, -S, \psi)$, or equivalently, a Bessel model of type $(\bar{\Lambda}, S, \psi^{-1})$, if there is a $\phi \in V_{\mathbf{F}}$ such that $B_{\bar{\phi}} \neq 0$. The archimedean component of the character ψ^{-1} coincides with the character we fixed in Section 2.5, so that our local theory applies without changes. Note also that, since $\pi_{\mathbf{F}}$ is irreducible, if $B_{\bar{\phi}} \neq 0$ for some $\phi \in V_{\mathbf{F}}$ then the same is true for all elements of $V_{\mathbf{F}}$. We now make the following important assumptions.

Assumption 1: $\pi_{\mathbf{F}}$ has a global Bessel model of type $(\bar{\Lambda}, S, \psi^{-1})$ such that $d(S) = -D$ is the fundamental discriminant of $\mathbb{Q}(\sqrt{-D})$.

Assumption 2: l is a multiple of $w(-D)$, the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$.

Remark. Assumption 1 implies that for any S' in any $\mathrm{SL}(2,\mathbb{Z})$ equivalence class of primitive semi-integral, positive definite matrices with $d(S') = -D$, we can find a Λ' such that $\pi_{\mathbf{F}}$ has a $(\Lambda', S', \psi^{-1})$ global Bessel model. This can be explained as follows. In [8, 1-26], Sugano has obtained the following formula for the vector-valued function $\mathbf{\Phi}$

defined in (115):

$$B_{\bar{\Phi}}(g_\infty) = \int_{Z(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})} (\Lambda \otimes \theta)(r)^{-1} \bar{\mathbf{\Phi}}(rg_\infty) dr$$

$$= \mu_2(g_\infty)^{l+(n-1)/2} \overline{\rho(CI+D)^{-1} e^{2\pi i \mathrm{tr}(S(g_\infty\langle I\rangle))} \pi_{\bar{\Lambda}} \left(\frac{1}{h(-D)} \sum_{i=1}^{h(-D)} \bar{\Lambda}^{-1}(u_i) \mathbf{A}(S_i) \right)}, \qquad (118)$$

where $g_\infty = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{GSp}(4,\mathbb{R})^+$, $\pi_\Lambda = \int_{T^1(\mathbb{R})} \rho(\zeta)^{-1} \Lambda_\infty^{-1}(\zeta) d^\times\zeta \in \mathrm{End}(V)$, $h(-D)$ is the class number of $\mathbb{Q}(\sqrt{-D})$, the elements $\{u_i\}$ are representatives of the idele class group of $\mathbb{Q}(\sqrt{-D})$, and $\{S_i\}$ is the orbit of S under the action of the u_i's. It is clear that $B_{\Psi(\bar{\Phi})} = \Psi(B_{\bar{\Phi}})$, and hence, one can find a Ψ such that $\pi_{\mathbf{F}}$ has a $(\bar{\Lambda}, S, \psi^{-1})$-Bessel model if and only if $B_{\bar{\Phi}} \neq 0$. Now, if $-D$ is the fundamental discriminant of $\mathbb{Q}(\sqrt{-D})$ then the set $\{S_i\}$ in (118) runs through all the $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes of primitive semi-integral, positive definite matrices with discriminant $-D$. This implies that the nonvanishing of $B_{\bar{\Phi}}$ depends only on $-D$ and Λ, not on the specific matrix S. Finally, note that (118) implies, in the scalar valued case, that Assumption 1 is equivalent to assuming that F has a nonzero Fourier coefficient $A(S)$, where S satisfies the condition from Assumption 1. □

From the above remark, it makes sense to consider

$$S(-D) = \begin{cases} \begin{bmatrix} \frac{D}{4} & 0 \\ 0 & 1 \end{bmatrix} & \text{if } D \equiv 0 \pmod 4, \\ \begin{bmatrix} \frac{1+D}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} & \text{if } D \equiv 3 \pmod 4. \end{cases} \qquad (119)$$

5.3 An integral representation

Let $N = \prod p^{n_p}$ be a positive integer. We denote the space of Maaß cusp forms of weight $l_1 \in \mathbb{Z}$ with respect to $\Gamma_0(N)$ by $S_{l_1}^M(N)$. A function $f \in S_{l_1}^M(N)$ has the Fourier expansion

$$f(x+iy) = \sum_{n\neq 0} a_n W_{\mathrm{sgn}(n)\frac{l_1}{2}, \frac{ir}{2}}(4\pi|n|y) e^{2\pi i n x}, \qquad (120)$$

where $W_{\nu,\mu}$ is a classical Whittaker function and $(\Delta_{l_1} + \lambda) f = 0$ with $\lambda = 1/4 + (r/2)^2$. Here Δ_{l_1} is the Laplace operator defined in Section 2.3 of [5]. Let $f \in S_{l_1}^M(N)$ be a Hecke eigenform. If $ir/2 = (l_2 - 1)/2$ for some integer $l_2 > 0$, then the cuspidal, automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ constructed below is holomorphic at infinity of lowest weight l_2. In this case we make the additional assumptions that $l_2 \le l$ and $l_2 \le l_1$, where l is coming from the Siegel cusp form $\mathbf{F}$ as in the previous section. Starting from a Hecke eigenform f, we obtain another Maaß form $f_l \in S_l^M(N)$ by applying the raising and lowering operators as in Section 2.3 of [5]. Define a function $\hat{f}$ on $\mathrm{GL}(2, \mathbb{A})$ by

$$\hat{f}(\gamma_0 m k_0) = \left(\frac{\gamma i + \delta}{|\gamma i + \delta|}\right)^{-l} f_l\left(\frac{\alpha i + \beta}{\gamma i + \delta}\right), \tag{121}$$

where $\gamma_0 \in \mathrm{GL}(2, \mathbb{Q})$, $m = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathbb{R})^+$, $k_0 \in \prod_{p|N} K^{(1)}(\mathfrak{p}^{n_p}) \prod_{p\nmid N} \mathrm{GL}(2, \mathbb{Z}_p)$. Here, for $p|N$ we have $K^{(1)}(\mathfrak{p}^{n_p}) = \mathrm{GL}(2, \mathbb{Q}_p) \cap \left[\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ \mathfrak{p}^{n_p} & \mathbb{Z}_p^\times \end{smallmatrix}\right]$ with $\mathfrak{p} = p\mathbb{Z}_p$. Let $\tau_f \cong \otimes'_p \tau_p$ be the irreducible, cuspidal, automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ generated by $\hat{f}$.

Let the unitary group $\mathrm{GU}(2, 2; L)(\mathbb{A})$ and its subgroups P, $M^{(1)}$, $M^{(2)}$, and N be as defined in pp. 190, 192 of [2] (or Section 2.1 of [6]). As in p. 210 of [2] (or Section 5.2 of [6]), define an Eisenstein series on $\mathrm{GU}(2, 2; L)(\mathbb{A})$ by

$$E_\Lambda(g, s) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{GU}(2,2;L)(\mathbb{Q})} f_\Lambda(\gamma g, s). \tag{122}$$

Here, f_Λ is the function on $\mathrm{GU}(2, 2; L)(\mathbb{A})$ defined by

$$f_\Lambda(mnk, s) = \delta_P^{\frac{1}{2}+s}(m) \Lambda(\bar{m}_1)^{-1} \hat{f}(m_2) \det(J(k_\infty, I))^{-l},$$

where $m = m_1 m_2$, $m_i \in M^{(i)}(\mathbb{A})$, $n \in N(\mathbb{A})$, and k is from the standard maximal compact subgroup (see p. 209 of [2]). For any vector $\phi \in V_{\mathbf{F}}$, we consider the global integral

$$Z(s, \Lambda) = \int_{Z(\mathbb{A})\mathrm{GSp}(4,\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{A})} E_\Lambda(h, s) \bar{\phi}(h) dh. \tag{123}$$

In Theorem 2.4 of [2], the following basic identity has been proved. Let $S = \left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right]$ and $\alpha = (b + \sqrt{-D})/(2c)$. Then

$$Z(s, \Lambda) = \int_{R(\mathbb{A})\backslash \mathrm{GSp}(4,\mathbb{A})} W_\Lambda(\eta h, s) B_{\bar{\phi}}(h) dh, \qquad \eta = \begin{bmatrix} 1 & & & \\ \alpha & 1 & & \\ & & 1 & -\bar{\alpha} \\ & & & 1 \end{bmatrix}. \tag{124}$$

Here,

$$W_\Lambda(g,s)=\int_{F\backslash\mathbb{A}} f_\Lambda\left(\begin{bmatrix}1&&&\\&1&&x\\&&1&\\&&&1\end{bmatrix}g,s\right)\psi(cx)\,dx$$

(see p. 196 of [2]), and $B_{\bar\phi}$ is as defined in (117). Let $\pi_{\mathbf{F}}$ be given by its $(\bar\Lambda, S(-D), \psi^{-1})$-Bessel model. Let $\phi=\otimes\phi_p\in V_{\mathbf{F}}$ be such that for $p<\infty$ we choose ϕ_p to be the unique spherical vector in π_p satisfying $\phi_p(1)=1$ and ϕ_∞ to be a vector in the one-dimensional K-type (l,l) in the lowest weight representation $\mathcal{E}(l,l-(n-1))$. Recall that we have assumed n to be odd, so that the one-dimensional K-type (l,l) really occurs in $\mathcal{E}(l,l-(n-1))$. Since ϕ is a pure tensor, the global zeta integral factors,

$$Z(s,\Lambda)=\prod_p Z_p(s,\Lambda)=\prod_p\int_{R(\mathbb{Q}_p)\backslash \mathrm{GSp}(4,\mathbb{Q}_p)} W_p(\eta h,s)B_{\bar\phi_p}(h)dh.$$

For $p<\infty$ the integral $Z_p(s,\Lambda)$ has been evaluated in [2] and [6]. We will now compute $Z_\infty(s,\Lambda)$. From Section 4.7 of [2], we have

$$Z_\infty(s)=\pi\int_{\mathbb{R}^\times}\int_1^\infty W_\infty\left(\eta\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&{}^tt_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix},s\right)$$
$$B_{\bar\phi_\infty}\left(\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&{}^tt_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix}\right)(\zeta-\zeta^{-3})\lambda^{-4}\,d\zeta\,d\lambda, \tag{125}$$

where $t_0\in \mathrm{GL}(2,\mathbb{R})^+$ is such that $T^1(\mathbb{R})=t_0\mathrm{SO}(2)t_0^{-1}$. Let us first consider the case $D\equiv 0$ mod 4, in which case $S(-D)=\left[\begin{smallmatrix}\frac{D}{4}&0\\0&1\end{smallmatrix}\right]$ and $t_0=\left[\begin{smallmatrix}2^{\frac12}D^{-\frac14}&\\&2^{-\frac12}D^{\frac14}\end{smallmatrix}\right]$. From Section 4.4 of [6], we have for $\lambda>0$

$$W_\infty\left(\eta\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&t_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix},s\right)$$
$$=c(1)\left|\lambda D^{-\frac12}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{-1}\right|^{3(s+\frac12)}W_{\frac{l}{2},\frac{ir}{2}}\left(4\pi\lambda D^{1/2}\frac{\zeta^2+\zeta^{-2}}{2}\right), \tag{126}$$

where $c(1)$ is the first Fourier coefficient of f_l. Using the results obtained in Section 5.1, we will state the formula for the Bessel function appearing in (125). Observe that $\Lambda_\infty \equiv 1$. By arguments from Section 2.6 on change of Bessel models, and the fact that π_{F} has trivial central character, we obtain

$$B_{\bar{\phi}_\infty}\left(\begin{bmatrix} \lambda t_0 \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & {}^t t_0^{-1} \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix}\right) = B_{l,l}\big(h(\lambda D^{\frac{1}{2}} 2^{-1}, \zeta, 0, 0)\big), \tag{127}$$

where $B_{l,l}$ is the vector in the one-dimensional K-type obtained in Section 5.1. We computed the precise formula for $B_{l,l}$ for $\dim(V) = n = 3, 5, 7, 9$. In each case, the formula has the shape

$$B_{l,l}\big(h(\lambda D^{\frac{1}{2}} 2^{-1}, \zeta, 0, 0)\big) = \left(\sum_{j=0}^{[\frac{n-1}{4}]} \sum_{k=2j}^{\frac{n-1}{2}} c_{k,j} \left(\frac{\lambda D^{\frac{1}{2}}}{2}\right)^{l-k} x^{k-2j}\right) e^{-4\pi\lambda \frac{D^{1/2}}{x}}, \tag{128}$$

where $x = (\zeta^2 + \zeta^{-2})/2$ and $c_{k,j}$ are real constants depending on k, j, l, n. Hence

$$Z_\infty(s) = \sum_{j=0}^{[\frac{n-1}{4}]} \sum_{k=2j}^{\frac{n-1}{2}} c_{k,j}\, Z_\infty^{k,j}(s),$$

where

$$\begin{aligned} Z_\infty^{k,j}(s) &= \pi \left(\frac{D^{\frac{1}{2}}}{2}\right)^{l-k} c(1) \int_0^\infty \int_1^\infty \lambda^{3(s+\frac{1}{2})+l-k} D^{-\frac{3}{2}(s+\frac{1}{2})} x^{-3(s+\frac{1}{2})+k-2j} \\ &\quad \times W_{\frac{l}{2},\frac{ir}{2}}(4\pi\lambda D^{\frac{1}{2}} x) e^{-2\pi\lambda D^{\frac{1}{2}} x} \lambda^{-4}\, dx\, d\lambda \\ &= \frac{c(1) 2^{-6s+3-3l+3k} D^{-3s} \pi^{-3s-l+k+\frac{5}{2}}}{6s+l-2k-1+2j} \frac{\Gamma\big(3s+l-k-1+\frac{ir}{2}\big)\Gamma\big(3s+l-k-1-\frac{ir}{2}\big)}{\Gamma\big(3s+\frac{l}{2}-k-\frac{1}{2}\big)} \\ &= Q_{k,j}(s) c(1) 2^{-6s+3-3l} D^{-3s} \pi^{-3s-l+\frac{5}{2}} \frac{\Gamma\big(3s+l-1+\frac{ir}{2}\big)\Gamma\big(3s+l-1-\frac{ir}{2}\big)}{\Gamma\big(3s+\frac{l}{2}-\frac{1}{2}\big)} \end{aligned}$$

with

$$Q_{k,j}(s) = \frac{2^{3k}\pi^k}{6s+l-2k-1+2j} \prod_{t=1}^{k} \frac{3s+\frac{l}{2}-\frac{1}{2}-t}{\big(3s+l-t-1+\frac{ir}{2}\big)\big(3s+l-t-1-\frac{ir}{2}\big)}.$$

Thus, we get

$$Z_\infty(s) = \left(\sum_{j=0}^{\left[\frac{n-1}{4}\right]} \sum_{k=2j}^{\frac{n-1}{2}} c_{k,j} Q_{k,j}(s) \right) c(1) 2^{-6s+3-3l} D^{-3s} \pi^{-3s-l+\frac{5}{2}} \frac{\Gamma\left(3s+l-1+\frac{ir}{2}\right)\Gamma\left(3s+l-1-\frac{ir}{2}\right)}{\Gamma\left(3s+\frac{l}{2}-\frac{1}{2}\right)}. \tag{129}$$

Recall that we did the above calculations under the assumption that $D \equiv 0 \pmod 4$. Following [2] or the methods from Section 4.4 of [6], we get the same formula for $Z_\infty(s)$ for $D \equiv 3 \pmod 4$. Using the non-archimedean calculations from [2] and [6] we now get the following global theorem.

Theorem 5.1. Let $\rho = \det^l \otimes \rho_0$, where l is an even positive integer and ρ_0 is the n-dimensional, irreducible representation of $\mathrm{PGL}(2,\mathbb{C})$ with $n = 3, 5, 7$, or 9. Let $\mathbf{F} \in S_\rho(\Gamma_2)$ be a cuspidal vector-valued Siegel eigenform of degree 2 satisfying the two assumptions in Section 5.2. Let $L = \mathbb{Q}(\sqrt{-D})$, where D is as in Assumption 1. Let $N = \prod p^{n_p}$ be a positive integer. Let f be a Maaß-Hecke eigenform of weight $l_1 \in \mathbb{Z}$ with respect to $\Gamma_0(N)$. If f lies in a holomorphic discrete series with lowest weight l_2, then assume that $l_2 \le l$. Then the integral (123) is given by

$$Z(s,\Lambda) = \kappa_N(s) Z_\infty(s) \frac{L\left(3s+\frac{1}{2}, \pi_{\mathbf{F}} \times \tau_f\right)}{\zeta(6s+1) L(3s+1, \tau_f \times \mathcal{AI}(\Lambda))}, \tag{130}$$

where

$$\kappa_N(s) = \prod_{p|N} \frac{p-1}{p^{3(n_p-1)}(p+1)(p^4-1)} \left(1 - \left(\frac{L}{p}\right) p^{-1}\right) p^{n_p} (1-p^{-6s-1})^{-1} \prod_{p^2|N} L_p(3s+1, \tau_p \times \mathcal{AI}(\Lambda_p)),$$

$$\left(\frac{L}{p}\right) = \begin{cases} -1 & \text{if } p \text{ is inert in } L, \\ 0 & \text{if } p \text{ ramifies in } L, \\ 1 & \text{if } p \text{ splits in } L. \end{cases}$$

$Z_\infty(s)$ is as in (129) and $\mathcal{AI}(\Lambda)$ is the representation of $\mathrm{GL}(2,\mathbb{A})$ obtained by automorphic induction from Λ. □

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